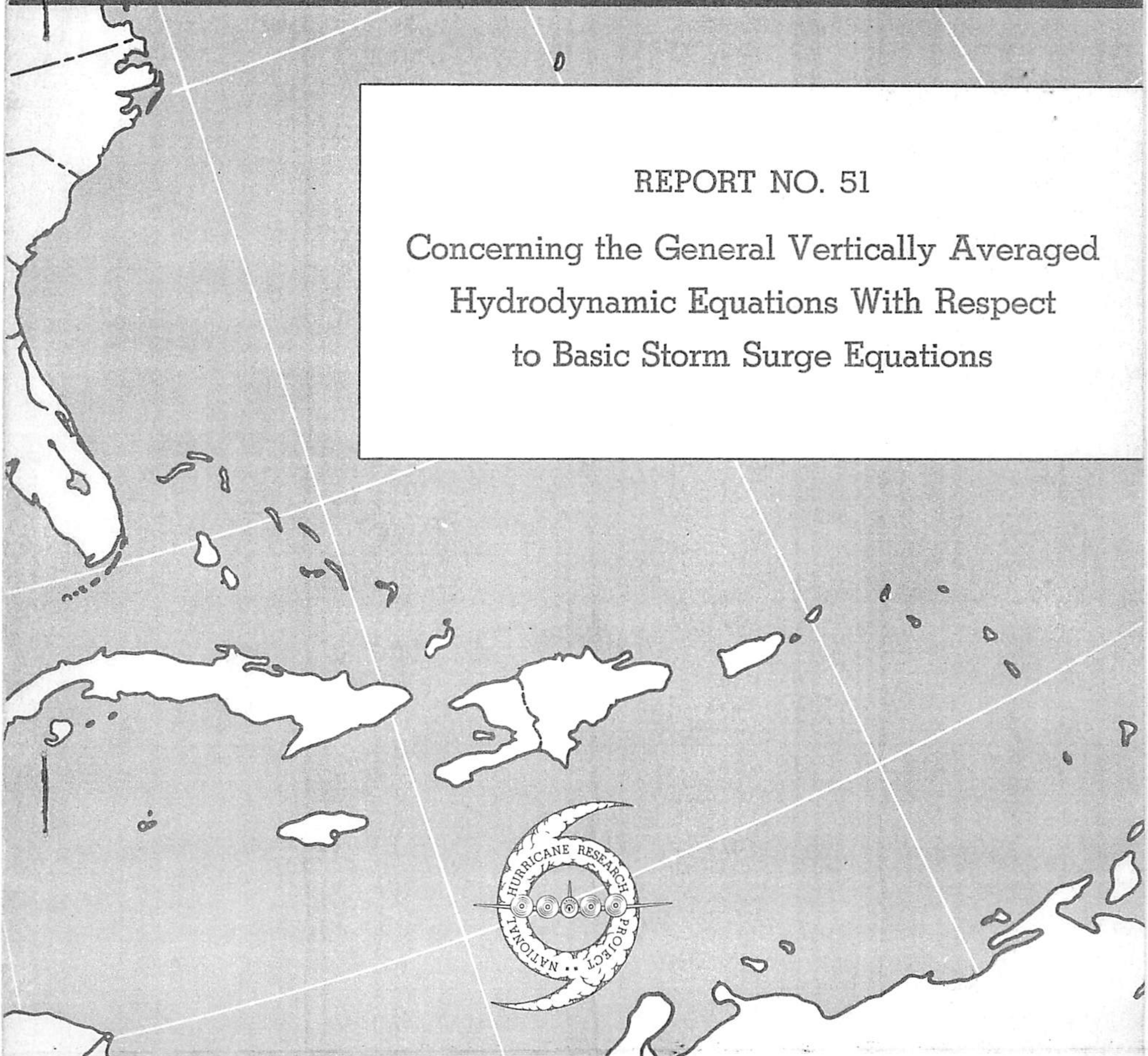
NATIONAL HURRICANE RESEARCH PROJECT

REPORT NO. 51

Concerning the General Vertically Averaged Hydrodynamic Equations With Respect to Basic Storm Surge Equations



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Concerning the General Vertically Averaged Hydrodynamic Equations With Respect to Basic Storm Surge Equations

by

Heinz G. Fortak
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NATIONAL HURRICANE RESEARCH PROJECT REPORTS

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CONTENTS

CONCERNING THE GENERAL VERTICALLY AVERAGED HYDRODYNAMIC EQUATIONS WITH RESPECT TO BASIC STORM SURGE EQUATIONS

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ABSTRACT

The three-dimensional hydrodynamic equations for an inhomogeneous fluid are used to derive a system of equations for the study of storm surges in an exceptionally complete form. Smoothing over a short time interval leads to the introduction of small-scale turbulent friction terms but guarantees a mathematically defined free surface of continuous slope. However, many of the characteristics of the wind-driven surface waves are retained.

Vertical averaging permits the introduction of the kinematic boundary conditions including the mass exchange at the surface. This also leads to the introduction of exact three-dimensional dynamic boundary conditions which contain all components of the turbulent stress tensor which act on the surface. Many terms which are usually neglected in linear theories cancel each other when the complete non-linear equations are integrated in the vertical.

A second process of time averaging is used to filter all surface waves below a certain wave period. This leads to an expression for the effect of the surface waves on sea level as averaged over a period of several minutes.

The final equations for the mean water transport contain the following effects which may be important in the development of storm surges: a moving atmospheric pressure field; moving field of tangential wind stresses; the wave set-up; horizontal and vertical density gradients in the free ocean or near the coast; non-hydrostatic terms which can be expressed partly by the kinetic

energy of the surface particles, and therefore by means of the kinematic boundary condition in terms of the equation of the surface itself, and additional surface stresses connected with the total wind stress at the surface and the total stresses at the bottom in the case in which the slope of the bottom is not small; and the precipitation-evaporation effect.

From this point it is possible to derive most basic equations used in storm surge investigations and in many related oceanographic problems in a manner which shows the approximations involved. The separation of a basic current, such as that which may arise from the astronomical tides or the general circulation of the ocean, from the complete equations shows that the storm surge cannot, in general, be treated by perturbation methods.

The equations are derived in vectorial form to permit the ready adoption of any special coordinate system suitable to a specific problem. Some potentially useful but unusual coordinate systems are discussed. Finally, vorticity and divergence equations are derived from the general equations for both the mean velocity and the mean mass transport. Several frequently used systems of equations can be derived from these expressions in a manner which shows the terms that are commonly neglected in the simplified theories.

1. INTRODUCTION

The basic equations used in most theoretical storm surge research as in most other modern research in dynamic oceanography are the vertically integrated hydrodynamic equations. This facilitates the introduction of the boundary conditions applicable to the vertical coordinate, and it is these boundary conditions that are dominant in most problems in physical oceanography.

This dominance of the upper and lower boundaries in oceanography is in contrast to meteorology where only the lower boundary is present and even this has relatively little effect on the character of the large-scale motions. Thus one of the essential differences between problems of dynamic oceanography and those of dynamic meteorology is given by the differences in the boundary conditions. The two boundary value problems of oceanography are, in principle, much more complicated than the one boundary value problem of meteorology.

The vertical integration of the hydrodynamic equations, usual in oceanography, transforms the difficulties of solving the three-dimensional boundary value problem into difficulties regarding the unknown stress and pressure at the bottom of the sea. The disadvantage of introducing these unknown functions is compensated for by several important advantages. In fact, satisfactory solutions of several oceanographic problems have been obtained by combining the vertically integrated equations with the hydrostatic approximation in rather simple expressions for the bottom stress, and even in fully linearized models (Sverdrup [31,32], Stommel [30], Hidaka [15, 16, 17], Munk [21], and others). These linearized models, which are much simpler than those for atmospheric motions, are applicable to many problems involving the oceanic currents.

The same simplified form of the basic equations has been used successfully in several storm surge research problems. However, there are other problems in storm surge prediction that are not solved from a theoretical point of view. There are many situations in which the quantitative predictions of tide height during a storm cannot be made with satisfactory accuracy. This is true, partly because the storm surge prediction requires a priori an accurate detailed prediction of the meteorological conditions during the storm (not presently available), and partly because the mathematical problem is more complicated than one might expect.

There are several difficulties associated with the migratory boundary during the inundation phase and non-linear interaction between the surge and the astronomical current (Doodson [7, 8], and Proudman [23,24]), and other non-linear terms which can generally be neglected in the deep ocean but which are significant in the vicinity of the coast. These problems cannot be solved by a simple addition of particular solutions obtained from a linear theory.

Attempts have been made to incorporate non-linear effects by using an extension of the equations for the first approximation of the non-linear shallow water wave theory (Freeman [10]) and the method of wave derivatives (Freeman and Baer [11, 13]). This latter method permits some investigation of the interactions between the surge and the basic currents.

In this paper the problem will be formulated anew on the basis of the general hydrodynamic equations. All approximations and assumptions will be deferred as long as possible with the hope of obtaining a better understanding of the effects of the terms which may have to be neglected before a quantitative answer to the problem as obtained. In many special cases the approximations which must ultimately be introduced could be introduced at the very beginning with a resultant simplification of the equations. However, this is not always true, and it is inefficient as each derivation with special assumptions must be repeated for each special problem. Special cases can be obtained directly from a general development of the mass transport equation by the introduction of suitable assumptions in the final equation. Moreover, when the derivation is carried out in this way, one has a better evaluation of the validity of the special equation being used.

The mass transport equations derived here are believed to contain, as special cases, all of the basic equations used so far in connection with the vertically integrated equations in oceanography and many of those used in meteorology. The form of these equations is given in a manner which permits physical interpretation and which can be handled formally in a mathematical sense. The form of the equations should suggest the introduction of new approximations to obtain a more complete formulation of the practical problems.

2. BASIC EQUATIONS OF MOTION

2.1 Symbols and definitions

$\ell = 2\,\omega \cos \phi$

$f = 2\,\omega \sin \phi$

ϕ = co-latitude

$\mathbf{i}, \mathbf{j}, \mathbf{k}$	: Orthogonal system of unit vectors
$\mathcal{E} = \mathcal{E}_h + \mathbf{k}\mathbf{k}$	: Unit tensor
$\mathcal{E}_h = \mathbf{i}\mathbf{i} + \mathbf{j}\mathbf{j}$	: Two-dimensional unit tensor
$\boldsymbol{\omega} = \frac{\ell}{2}\mathbf{j} + \frac{f}{2}\mathbf{k}$	: Angular velocity vector of the earth rotation
$\nabla = \nabla_h + \mathbf{k}\frac{\partial}{\partial z}$	: Three-dimensional nabla-operator
$\mathbf{V} = \mathbf{V}_h + \mathbf{k} v_z$	: Three-dimensional velocity vector
$\mathbf{V}_h = \mathbf{i} v_x + \mathbf{j} v_y$	: Horizontal velocity vector
ρ, p	: Density and pressure
$\Phi^{(g)}$, g	: Gravity potential and gravity acceleration
$\Phi^{(T)}$, $\eta^{(T)}(x,y,t)$	: Astronomical tidal potential and equilibrium tide elevation

$\bar{\eta}(x,y,t)$	:	Mean free surface elevation after first time averaging.
$\mathcal{F}_{visc}$	:	Stress tensor of molecular friction
$\rho\mathbf{V}$	:	Linear momentum
$\rho\mathbf{V}\,\mathbf{V}$	:	Momentum transfer tensor
$\bar{A} = \frac{1}{T}\int_0^T A\,dt$	:	First time average of A
$A' = A - \bar{A}$		
$\hat{A} = \frac{1}{\bar{\rho}\,T}\int_0^T \rho A dt$	:	First weighted time average of A
$A'' = A - \hat{A}$		
$\overline{A'} = 0;\ \overline{\rho A''} = 0$		
$\mathcal{F} = -\overline{\rho\mathbf{V}''\mathbf{V}''}$	:	Reynolds-stress tensor (internal turbulent friction)

The following relations and notations are also useful:

$$\nabla\cdot\left|-p\mathcal{E}\right| = -\nabla p \tag{1}$$

$$\rho\frac{d\mathbf{V}}{dt} = \frac{\partial\rho\mathbf{V}}{\partial t} + \nabla\bullet\left|\rho\mathbf{V}\,\mathbf{V}\right| \tag{2}$$

$$\mathcal{F} = (-\overline{\rho\mathbf{V}''\mathbf{V}''}) = (-\overline{\rho\mathbf{V}''\mathbf{V}''_h}) + (-\overline{\rho\mathbf{V}''v''_z})\,\mathbf{k} = \mathcal{F}_h + \mathcal{F}_v \tag{3}$$

where

$$\begin{aligned}\mathcal{F}_h &= (-\overline{\rho\mathbf{V}''\mathbf{V}''_h}) = \mathbf{i}(-\overline{\rho v''_x\mathbf{V}''_h}) + \mathbf{j}(-\overline{\rho v''_y\mathbf{V}''_h}) + \mathbf{k}(-\overline{\rho v''_z\mathbf{V}''_h}) \\ &= \mathbf{i}\mathcal{T}_{x,h} + \mathbf{j}\mathcal{T}_{y,h} + \mathbf{k}\mathcal{T}_{z,h}\end{aligned} \tag{4}$$

the "horizontal" component of $\mathcal{F}$, and

$$\mathcal{F}_v = (-\overline{\rho\mathbf{V}''v''_z})\,\mathbf{k} = \mathcal{T}_z\,\mathbf{k} \tag{5}$$

the vertical components of $\mathcal{F}$.

2.2 Time averaged equations for linear momentum

The complete equations for linear momentum describing the turbulent motion of the fluid, and for conservation of mass are given by Eliassen and Kleinschmidt [9]:

$$\frac{\partial\rho\mathbf{V}}{\partial t} + \nabla\cdot\left|\rho\mathbf{V}\,\mathbf{V}\right| + 2\boldsymbol{\omega}\times\rho\mathbf{V} = -\rho\nabla\left|\Phi^{(g)} + \Phi^{(T)}\right| + \nabla\bullet\left|-p\mathcal{E} + \mathcal{F}_{visc}\right| \tag{6}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{V} = 0 \qquad (7)$$

The surface of the sea, when under the influence of a severe storm, cannot be described exactly by a single-valued function of position and time. A turbulent zone in which the air contains much spray and the water many air bubbles exists between the two fluids. A mathematically defined smooth surface, necessary for the vertical integration, which will be performed later, is obtained by taking a time average over a short period of the order of a fraction of a second to a few seconds. This averaging process introduces additional terms representing small-scale turbulent friction.

Using the definitions of ordinary and weighted mean given in 2.1, averaging of equations (6) and (7) leads to

$$\frac{\partial\bar{\rho}\hat{\mathbf{V}}}{\partial t} + \nabla\cdot\left\{\bar{\rho}\hat{\mathbf{V}}\hat{\mathbf{V}}\right\} + 2\boldsymbol{\omega}\times\bar{\rho}\hat{\mathbf{V}} = -\bar{\rho}\nabla\left\{\Phi^{(g)}+\Phi^{(T)}\right\} + \nabla\cdot\left\{-\bar{p}\mathcal{E}-\overline{\rho\mathbf{V}''\mathbf{V}''}+\bar{\mathcal{F}}_{visc}\right\} \qquad (8)$$

$$\frac{\partial\bar{\rho}}{\partial t} + \nabla\cdot\left\{\bar{\rho}\hat{\mathbf{V}}\right\} = 0. \qquad (9)$$

If the horizontal x, y plane is chosen to coincide with the surface $\Phi^{(g)} = 0$ and the vertical unit vector $\mathbf{k}$ is directed upwards, $\Phi^{(g)} = gz$ and $\nabla\Phi^{(g)} = g\mathbf{k}$. The tidal potential may be expressed as the equilibrium water elevation, $\eta^{(T)}$ (Defant [6]). By using these conventions and neglecting the molecular viscosity which is generally several orders of magnitude less than the turbulent viscosity, (8) can be written in the form

$$\frac{\partial\bar{\rho}\hat{\mathbf{V}}}{\partial t} + \nabla\cdot\left\{\bar{\rho}\hat{\mathbf{V}}\hat{\mathbf{V}}\right\} + 2\boldsymbol{\omega}\times\bar{\rho}\hat{\mathbf{V}} = -g\bar{\rho}\left\{\mathbf{k}-\nabla_h\eta^{(T)}\right\} + \nabla\cdot\left\{-\bar{p}\mathcal{E}-\overline{\rho\mathbf{V}''\mathbf{V}''}\right\} \qquad (10)$$

It is assumed that this averaging process guarantees a surface which is a single-valued function of time and position and which has finite slope, say $\bar{\eta} = \bar{\eta}(x,y,t)$.

Applying these equations to the motions of incompressible but possibly inhomogeneous water, as can be assumed in the case of storm surge calculations, the condition for incompressibility is

$$\frac{\partial\bar{\rho}}{\partial t} + \hat{\mathbf{V}}\cdot\nabla\bar{\rho} = 0$$

and with this condition equation (9) leads to the well-known equation of continuity for incompressible fluids

$$\nabla\cdot\hat{\mathbf{V}} = 0 \qquad (9')$$

We use, however, the general form (9) in the following derivations. This is more convenient and simplifies the resulting equations in some certain sense.

3. BOUNDARY CONDITIONS

3.1 Symbols and definitions

It will be helpful to refer to figure 1 for the interpretation of the following symbols.

$\overline{\Psi}_s = z - \overline{\eta}(x,y,t) = 0$: Mathematical equation of the surface (smoothed)

$\Psi_b = -\big(z + H(x,y,t)\big) = 0$: Equation of the bottom

$-(\overline{\Psi}_s + \Psi_b) = \overline{\eta} + H = 0$: Actual coast line on a sloping beach

$$\overline{n}_s = \frac{\nabla \overline{\Psi}_s}{|\nabla \overline{\Psi}_s|} = \frac{k - \nabla_h \overline{\eta}}{\sqrt{1 + |\nabla_h \overline{\eta}|^2}}$$: Surface unit normal vector

$$\overline{N}_s = k - \nabla_h \overline{\eta} = \frac{\overline{n}_s}{\cos \alpha_s}$$: Surface normal vector

$$\overline{n}_s \cdot \overline{C}_s = -\frac{\dfrac{\partial \overline{\Psi}_s}{\partial t}}{|\nabla \overline{\Psi}_s|} = \frac{\partial \overline{\eta}/\partial t}{\sqrt{1 + |\nabla_h \overline{\eta}|^2}}$$: Normal velocity of the surface

$$n_b = \frac{\nabla \Psi_b}{|\nabla \Psi_b|} = -\frac{k + \nabla_h H}{\sqrt{1 + |\nabla_h H|^2}}$$: Unit normal vector of the bottom

$$N_b = -(k + \nabla_h H) = \frac{n_b}{\cos \alpha_b}$$: Normal vector of the bottom

$$n_b \cdot C_b = -\frac{\dfrac{\partial \Psi_b}{\partial t}}{|\nabla \Psi_b|} = \frac{\partial H/\partial t}{\sqrt{1 + |\nabla_h H|^2}}$$: Normal velocity of the bottom

α_s, α_b : Slope of the surface and the bottom respectively

$\Delta \overline{m}$: Mass exchange at the surface (normal)

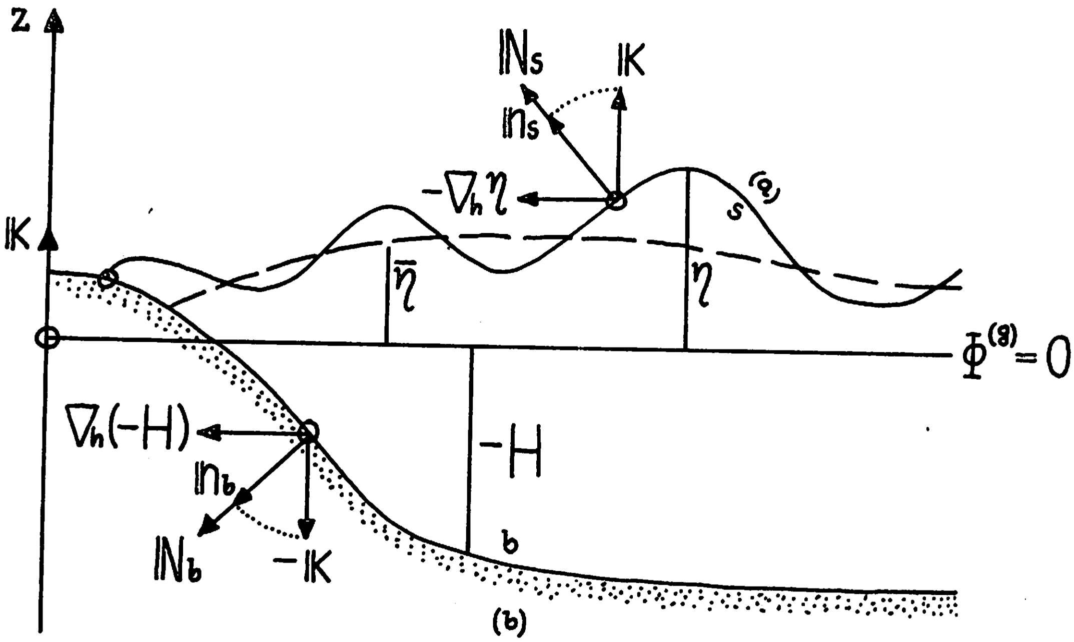
$\overline{P} - \overline{E} = \dfrac{\Delta \overline{m}}{\cos \alpha_s}$: Difference between precipitation and dynamic evaporation

Lower indices s, b : Describing properties of the water at the boundaries

Upper indices (a), (b) : Describing properties of the adjacent "fluids"

$\overline{\mathcal{S}} = -\overline{p}\,\mathcal{E} + \mathcal{F}$: Stress tensor

$n \cdot \overline{\mathcal{S}}$: Stress vector



$$\mathbb{N}_s = \mathbb{K} - \nabla_h \eta \qquad \mathbb{N}_b = -\mathbb{K} - \nabla_h H$$: normal vectors

$$\mathbb{n}_s = \frac{\mathbb{N}_s}{|\mathbb{N}_s|} \qquad \mathbb{n}_b = \frac{\mathbb{N}_b}{|\mathbb{N}_b|}$$: unit normal vectors

$$C_{n,s} = \frac{\frac{\partial \eta}{\partial t}}{|\mathbb{N}_s|} \qquad C_{n,b} = \frac{\frac{\partial H}{\partial t}}{|\mathbb{N}_b|}$$: normal velocities of the boundaries

Figure 1. - Explanation of the symbols.

3.2 General remarks

The kinematic and dynamic boundary conditions at surfaces of discontinuity require mathematically defined boundaries, internal or external, which possess derivatives of the first order. This may be guaranteed by the foregoing process of first time averaging, e.g. $\bar{\eta}$ (x,y,t) if the derivatives $\nabla_h \bar{\eta}$, $\partial\bar{\eta}/\partial t$ are considered to exist everywhere.

If the fluid consists of two or more dissimilar layers separated by surfaces of discontinuity the kinematic equation of discontinuity (9) must be replaced by a kinematic boundary condition and the dynamic equation of motion (10) must be replaced by a dynamic boundary condition (Serrin [27]). Both the upper and lower boundaries of the sea may be considered as surfaces of discontinuity.

The kinematic boundary condition at a discontinuity states that the surface divergence of the velocity at the boundary is proportional to the mass exchange at the boundary (Truesdell and Toupin [34]). If there is no mass exchange at the boundary, as generally assumed, this condition means that the normal components of the velocities on each side of the discontinuity surface are equal to the normal velocity of the surface itself. Storm surges are generally associated with severe storms and it is likely that in some regions locally heavy rain or surface run-off from rain occurring over land contribute a measurable part of the surge. The usual form of the kinematic boundary condition will not be satisfactory in such cases.

The dynamic boundary condition at a discontinuity states that if surface tension is absent the stress vector will be continuous at the discontinuity, and in the presence of surface tension the component of the stress normal to the boundary will have a discontinuity proportional to the mean curvature of the boundary surface (Serrin [27], Wehausen and Laitone [38]). This latter condition is usually neglected, but the work of Keulegan [19], Van Dorn [35] and others showing that the denevaluation produced by a given wind is decreased by the addition of detergent to the water imply that this assumption should be reexamined.

3.3 Kinematic boundary conditions

No significant mass exchange occurs at internal discontinuities or at the bottom. Consequently, at these discontinuities the surface divergence of the velocity vanishes. A mass exchange may take place across the interface between air and water. Normal evaporation is universally present. Spray is added to the air by the breaking of waves and by other processes. Some of this water evaporates before the resultant drops fall back to the sea surface. Water is added to the sea by precipitation, and near the horizontal boundaries of the sea, water is added by the run-off of rain water from land surfaces. All of these effects, other than precipitation and run-off, are due to dynamic effects near the surface of the sea. They may be considered collectively as "dynamic evaporation," positive when the sea is losing mass at the surface and negative when the sea is gaining mass. Therefore, at the surface a source or sink of mass occurs corresponding to the sign of the difference between precipitation and "evaporation."

Disregarding internal discontinuities the kinematic boundary conditions at the surface and at the bottom are

$$(\mathrm{Div}\,\hat{\mathbf{v}})_s = \bar{\mathbf{n}}_s \cdot \left\{ \mathbf{v}^{(a)} - \hat{\mathbf{v}}_s \right\} = \frac{\Delta\bar{m}}{\bar{\rho}_s} \tag{11}$$

$$(\mathrm{Div}\,\hat{\mathbf{v}})_b = \mathbf{n}_b \cdot \left\{ \mathbf{v}^{(b)} - \hat{\mathbf{v}}_b \right\} = 0. \tag{12}$$

Introducing the normal velocities of the boundary surfaces which replace the values $\bar{\mathbf{n}}_s \cdot \mathbf{v}^{(a)}$ and $\mathbf{n}_b \cdot \mathbf{v}^{(b)}$ respectively one obtains

$$\frac{\partial\bar{\eta}}{\partial t} - \bar{\mathbf{N}}_s \cdot \hat{\mathbf{v}}_s = \frac{\partial\bar{\eta}}{\partial t} + \hat{\mathbf{v}}_{h,s} \cdot \nabla_h\bar{\eta} - \hat{v}_{z,s} = \frac{\Delta\bar{m}}{\bar{\rho}_s}\sqrt{1 + |\nabla_h\bar{\eta}|^2} = \frac{\Delta\bar{m}}{\bar{\rho}_s \cos\alpha_s} \tag{13}$$

$$\frac{\partial H}{\partial t} - \mathbf{N}_b \cdot \hat{\mathbf{v}}_b = \frac{\partial H}{\partial t} + \hat{\mathbf{v}}_{h,b} \cdot \nabla_h H + \hat{v}_{z,b} = 0 \tag{14}$$

or

$$\frac{\partial \bar{\eta}}{\partial t} - \overline{\mathbf{N}}_s \cdot \hat{\mathbf{v}}_s = \frac{1}{\rho_s} (\bar{P} - \bar{E}) ; \quad z = \bar{\eta}(x,y,t) \tag{15}$$

$$\frac{\partial H}{\partial t} - \mathbf{N}_b \cdot \hat{\mathbf{v}}_b = 0 \qquad ; \quad z = - H(x,y,t) \tag{16}$$

The kinematic boundary condition at the actual coast with a sloping bottom, $\bar{\eta}+H = 0$, can be formulated as

$$\frac{\partial(\bar{\eta} + H)}{\partial t} + \hat{\mathbf{v}}_{h,coast} \cdot \nabla_h(\bar{\eta} + H) = \frac{1}{\rho_s} (\bar{P} - \bar{E}); \ (\bar{\eta}+H) = 0 \tag{17}$$

The kinematic boundary condition for a vertical coast or seawall is usually expressed as

$$\mathbf{n}_b \cdot \hat{\mathbf{v}} = 0. \tag{18}$$

A general time dependent equation for the bottom has been used for the sake of generality. This causes no difficulties in the following considerations and may permit application of the resulting equations to Tsunami problems.

3.4 Dynamic boundary conditions

In the absence of surface tension the dynamic boundary condition at interfaces between dissimilar fluids is given by the vanishing surface divergence of the stress vector. The boundary between water and bottom material may be considered as an interface between two dissimilar fluids. Thus the condition of vanishing surface divergence of the stress vector (Serrin [27]) means

$$(\mathrm{Div}\, \mathcal{S})_s = \bar{\mathbf{n}}_s \cdot (\mathcal{S}^{(a)} - \mathcal{S}_s) = 0 \tag{19}$$

$$(\mathrm{Div}\, \mathcal{S})_b = \mathbf{n}_b \cdot (\mathcal{S}^{(b)} - \mathcal{S}_b) = 0 \tag{20}$$

This can be written by means of the definitions of the stress vector and the normal vectors of the surfaces

$$\overline{\mathbf{N}}_s \cdot (- \bar{p}_s \mathcal{E} + \mathcal{F}_s) = \overline{\mathbf{N}}_s \cdot (- \bar{p}^{(a)} \mathcal{E} + \mathcal{F}^{(a)}) \tag{21}$$

$$\mathbf{N}_b \cdot (- \bar{p}_b \mathcal{E} + \mathcal{F}_b) = \mathbf{N}_b \cdot (- \bar{p}^{(b)} \mathcal{E} + \mathcal{F}^{(b)}) \tag{22}$$

In the case of surface tension (19) has to be replaced by

$$(\mathrm{Div}\, \mathcal{S})_s = \mathbf{n}_s \cdot (\mathcal{S}^{(a)} - \mathcal{S}_s) = \frac{T}{R} \mathbf{n}_s \tag{19'}$$

where T is the coefficient of surface tension and R is the radius of curvature of the surface (Wehausen and Laitone [34]). In this case (21) can be written as

$$\overline{\mathbf{N}}_s \cdot (-\overline{p}_s \mathcal{E} + \mathcal{F}_s) = \overline{\mathbf{N}}_s \cdot \left\{ -(\overline{p}^{(a)} + \frac{T}{R}) \mathcal{E} + \mathcal{F}^{(a)} \right\} \tag{21'}$$

This shows, that surface tension provides a correction of the external atmospheric pressure $\overline{p}^{(a)}$.

Therefore, in the following sections surface tension can be included easily by replacing $\overline{p}^{(a)}$ by $\overline{p}^{(a)} + \frac{T}{R}$, where the radius, R, of curvature can be expressed in terms of derivations of the equation, $\overline{\eta}$, of the surface.

In principle, the storm surge could be calculated by solving equations (9) and (10) together with the kinematic boundary conditions (15) and (16) and the dynamic boundary conditions (21) and (22). Unfortunately, this is not possible even in the simplest cases.

A remarkable simplification is possible if the boundary condition (22) is known at the beginning of the calculation. However, this boundary condition depends on the pressure at the bottom and on the bottom stresses and these properties cannot be known before solving the complete problem. The usual practice is to assume that the hydrostatic law holds for the pressure at the bottom. This appears to be valid in most cases but it cannot be entirely valid for a consideration of the effect of surface waves on the water elevation in the neighborhood of the breaker zone. Even when the assumption of the hydrostatic law for the bottom pressure is acceptable there remains the problem of the stress vector $\mathcal{T}_{z,h}$.

The classical method around this impasse, first used by Stokes in the nineteenth century, is to apply a no slip or adherence condition defined by $\Delta \mathbf{V} = 0$, where $\Delta \mathbf{V}$ denotes the difference between the velocity of the fluid and that of the boundary material. This has been widely used and has led to satisfactory results in several problems. The more recent work on boundary layer theory indicates that the no slip conditions cannot be rigorous as a description of nature and most later writers employ some type of no slip condition, e.g.

$$\Delta \mathbf{V} \backsim f(\mathbf{n} \cdot \mathfrak{S}) \text{ tang.}$$

A linear relationship between $\Delta \mathbf{V}$ and the bottom stress is generally used when a qualitative but analytic solution is sought. A quadratic relation between $\Delta \mathbf{V}$ and stress is generally employed when quantitative results are sought and one is reasonably confident that he knows the proper direction of the bottom stress vector. The no slip condition is sometimes used for quantitative investigations if one has no <u>a priori</u> knowledge of the direction of the bottom stress vector, in spite of defects in the physical theory, because this will minimize the errors resulting from this lack of knowledge. The no slip condition can be stated explicitly as

$$V_b = \frac{\frac{\partial H}{\partial t}}{1 + |\nabla_h H|^2} N_b \tag{23}$$

thus expressing the velocity of the fluid bottom in terms of the motion of the rigid bottom. The case for the stationary bottom is included for $\partial H/\partial t = 0$. Other plausible assumptions regarding the stress vector are discussed in many recent papers, e.g., Weenink [37], Schalkwijk [26], Reid [25], and Munk [21], in connection with vertically integrated forms of equations (9) and (10).

The current, more rigorous treatment, requires not only the stress vector $\mathcal{T}_{z,h}$ but also the stress vectors $\mathcal{T}_{x,h}$ and $\mathcal{T}_{y,h}$ which are important if the slope of the bottom is not small. The same considerations hold for the surface condition (21) where all components of the stress vector including those which act on the waves of finite height are represented.

Principally because of the unknown bottom stresses, the use of the vertically integrated form of equations (9) and (10) cannot provide the ultimate solution of the storm surge problem. This must be kept in mind even if many investigations show that certain special assumptions concerning the bottom stresses give good numerical results for particular problems. However, as already mentioned, the advantages of using the vertically integrated equations of motion in many oceanographic problems are so great that the disadvantage incurred by the unknown bottom stresses is less than the advantages gained by this method.

4. VERTICALLY AVERAGED EQUATIONS

4.1 Symbols and definitions

$\overset{\text{---}}{A} = \frac{1}{\bar{\eta} + H} \int_{-H}^{\bar{\eta}} A dz$: Vertical mean of a function A(x,y,z,t)

$\hat{A} = \frac{1}{\overset{\text{---}}{\bar{\rho}}(\bar{\eta} + H)} \int_{-H}^{\bar{\eta}} \bar{\rho} A dz$: Weighted vertical mean of a function A(x, y,z,t)

$A = \overset{\text{---}}{A} + A^* = \hat{A} + A^{**}$: Definition of ordinary and weighted vertical mean

$\overset{\text{---}}{A^*} = 0; \quad \overline{\bar{\rho}}\, \overset{\text{---}}{A^{**}} = 0$: Properties of ordinary and weighted mean

$\int_{-H}^{\bar{\eta}} \bar{\rho} dz = (\bar{\eta} + H) \overset{\text{---}}{\bar{\rho}} = Q$: Instantaneous mass of a vertical column

$\int_{-H}^{\bar{\eta}} \bar{\rho} \hat{\mathbf{V}} dz = (\bar{\eta} + H) \overset{\text{---}}{\bar{\rho}} \hat{\hat{\mathbf{V}}} = Q \mathbb{V}$: Instantaneous mass transport vector

$-\overline{\bar{\rho}\hat{V}^{**}\hat{V}^{**}}$: Reynolds' stresses resulting from vertical averaging

$-(\overline{\rho V''V''})^{(a)}$: Surface wind stresses

4.2 Rules for vertical averaging of derivatives

Taking into account that the limits $\bar{\eta}(x,y,t)$ and $H(x,y,t)$ of the integral in the averaging operator depends on the horizontal coordinates and the time, the following rules for averaging of derivatives can be derived easily:

$$
\begin{aligned}
(\bar{\eta}+H)\overline{\frac{\partial A}{\partial t}} &= \frac{\partial}{\partial t}\left\{(\bar{\eta}+H)\bar{A}\right\} - \frac{\partial \bar{\eta}}{\partial t}A_s - \frac{\partial H}{\partial t}A_b \\
(\bar{\eta}+H)\overline{\nabla_h A} &= \nabla_h\left\{(\bar{\eta}+H)\bar{A}\right\} - (\nabla_h\bar{\eta})A_s - (\nabla_h H)A_b \\
(\bar{\eta}+H)\overline{\nabla A} &= \nabla_h\left\{(\bar{\eta}+H)\bar{A}\right\} + \bar{N}_s A_s + N_b A_b
\end{aligned}
\tag{24}
$$

for any scalar or vector function. This can also be written

$$
\begin{aligned}
\overline{\frac{\partial A}{\partial t}} &= \frac{\partial \bar{A}}{\partial t} + \frac{1}{\bar{\eta}+H}\left\{\frac{\partial\bar{\eta}}{\partial t}(\bar{A}-A_s) + \frac{\partial H}{\partial t}(\bar{A}-A_b)\right\} \\
\overline{\nabla_h A} &= \nabla_h\bar{A} + \frac{1}{\bar{\eta}+H}\left\{(\nabla_h\bar{\eta})(\bar{A}-A_s) + (\nabla_h H)(\bar{A}-A_b)\right\} \\
\overline{\nabla A} &= \nabla_h\bar{A} + \frac{1}{\bar{\eta}+H}\left\{\nabla_h(\bar{\eta}+H)\bar{A} + \bar{N}_s A_s + N_b A_b\right\}
\end{aligned}
\tag{25}
$$

4.3 Vertically averaged equations

Applying (24) to equations (9) and (10) one obtains

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left\{(\bar{\eta}+H)\overline{\bar{\rho}\hat{V}}\right\} + \nabla_h\cdot\left\{(\bar{\eta}+H)\overline{\bar{\rho}\hat{V}_h\hat{V}}\right\} - \left\{\frac{\partial\bar{\eta}}{\partial t} - \bar{N}_s\cdot\hat{V}_s\right\}(\bar{\rho}\hat{V})_s - \left\{\frac{\partial H}{\partial t} - N_b\cdot\hat{V}_b\right\}(\bar{\rho}\hat{V})_b \\
&\quad + 2\omega\times(\bar{\eta}+H)\overline{\bar{\rho}\hat{V}} = -g(\bar{\eta}+H)\overline{\bar{\rho}}\left\{k - \nabla_h\eta^{(T)}\right\} + \nabla_h\cdot\left\{(\bar{\eta}+H)(-\overline{\bar{p}\mathcal{E}_h} - \overline{\rho V_h''V''})\right\} \\
&\quad + \bar{N}_s\cdot\left\{-\bar{p}_s\mathcal{E} - (\overline{\rho V''V''})_s\right\} + N_b\cdot\left\{-\bar{p}_b\mathcal{E} - (\overline{\rho V''V''})_b\right\}.
\end{aligned}
$$

$$
\frac{\partial}{\partial t}\left\{(\bar{\eta}+H)\bar{\bar{\rho}}\right\} + \nabla_h\cdot\left\{(\bar{\eta}+H)\overline{\bar{\rho}\hat{V}_h}\right\} - \left\{\frac{\partial\bar{\eta}}{\partial t} - \bar{N}_s\cdot\hat{V}_s\right\}\rho_s - \left\{\frac{\partial H}{\partial t} - N_b\cdot\hat{V}_b\right\}\rho_b = 0
$$

Introducing the kinematic and dynamic boundary conditions (15), (16), and (21) together with the definitions of ordinary and weighted means given in 4.1, the vertically averaged equations are

$$\frac{\partial}{\partial t}\left[Q\mathbb{V}\right] + \nabla_h \cdot \left\{ Q\mathbb{V}_h\mathbb{V}\right\} + 2\boldsymbol{\omega} \times Q\mathbb{V} =$$

$$= - gQ\left\{\mathbf{k} - \nabla_h\eta^{(T)}\right\} + \nabla_h \cdot \left\{(\bar{\eta}+H)\left(-\bar{\bar{p}}\,\mathcal{E}_h - \overline{\overline{\rho \mathbf{V}_h''\mathbf{V}''}} - \overline{\overline{\bar{\rho}\hat{\mathbf{V}}_h^{**}\hat{\mathbf{V}}^{**}}}\right)\right\} \tag{26}$$

$$+ \overline{\mathbf{N}}_s \cdot \left\{-\bar{p}^{(a)}\mathcal{E} - (\overline{\rho\mathbf{V}''\mathbf{V}''})^{(a)}\right\} + \mathbf{N}_b \cdot \left\{-\bar{p}_b\mathcal{E} - (\overline{\rho\mathbf{V}''\mathbf{V}''})_b\right\} + (\bar{P}-\bar{E})\hat{\mathbf{V}}_s.$$

$$\frac{\partial Q}{\partial t} + \nabla_h \cdot \left\{Q\mathbb{V}_h\right\} = \bar{P}-\bar{E}, \tag{27}$$

where only the stress at the surface assumed to be known has been introduced.

These equations for the field variables $Q, \mathbb{V}$ are quite similar to the original equations (10) and (9) for $\bar{\rho}$ and $\hat{\mathbf{V}}$ with the exception that the vertical divergence terms are replaced by terms representing sources or sinks of the mass transport vector $Q\mathbb{V}$; e.g. the additional friction term resulting from vertical averaging and the stresses at the boundaries together with the precipitation minus evaporation effect.

At this point it may be noticed that so far no assumption regarding the heights or the shapes of the surface waves has been introduced. The process of vertical averaging has been performed exactly with the aid of the complete kinematic and dynamic boundary conditions. This was possible only because the complete non-linear equations of motion were used. Consequently no terms needed to be "neglected" as is the case using the well-known linearized equations. Many of the terms usually neglected cancel each other exactly.

Unfortunately the process of vertical averaging introduces new unknown functions, as already mentioned. The system (9), (10) contains the functions $\hat{\mathbf{V}}$, $\bar{\rho}$ and $\bar{p}$ whereas the system (26), (27) contains $\mathbb{V}$, $\bar{\bar{\rho}}$, and $\bar{\bar{p}}$ together with $\bar{\eta}$ and the unknown bottom pressure and bottom stress, i.e. $\bar{p}_b$, $\mathbf{N}_b \cdot (\overline{-\rho\mathbf{V}''\mathbf{V}''})_b$. In addition, the turbulent friction stress tensor $-\overline{\overline{\bar{\rho}\hat{\mathbf{V}}^{**}\hat{\mathbf{V}}^{**}}}$ resulting from vertical averaging remains an unknown function. The system (9), (10) can be made a closed one by assuming barotropy or by assuming a given density distribution. The system (26), (27) cannot be made a closed one without using very restrictive assumptions regarding the calculation of pressure within the fluid and at the bottom, and regarding the nature of the stresses at the bottom, Usually these difficulties are reduced to the problem of calculating the bottom stresses by assuming hydrostatic equilibrium and by introducing a given density distribution taken from observations. In such case the system is a closed one but since the hydrostatic approximation acts as a kind of low pass filter, filtering out all wave periods up to a certain period, leaving only so-called long waves, all effects connected with the superimposed small waves which may be important under hurricane conditions and which are certainly important during the inundation phase, are neglected. Therefore, it might be more convenient to perform a second process of time averaging in order to get equations for the long wave-type hurricane surge leaving only some correlation products between the superimposed small surface waves and other properties and then using for the surge equations the hydrostatic approximation.

One might think that it would be simpler, when dealing with storm surges, to begin with the equation of continuity for incompressible fluids (9'). Actually, it is simpler to perform the vertical integration on the general continuity equation and to introduce the incompressibility assumption at a later stage. In order to show this we apply the process of vertical averaging on the equation of continuity (9') for incompressible fluids which gives

$$(\bar{\eta}+H)\,\overline{\nabla\cdot\hat{\mathbf{v}}} = \nabla_h\cdot\left\{(\bar{\eta}+H)\,\overline{\hat{\mathbf{v}}_h}\right\} + \mathbf{N}_s\cdot\hat{\mathbf{v}}_s + \mathbf{N}_b\cdot\hat{\mathbf{v}}_b = 0$$

or with (15), (16)

$$\frac{\partial(\bar{\eta}+H)}{\partial t} + \nabla_h\cdot\left\{(\bar{\eta}+H)\,\overline{\hat{\mathbf{v}}_h}\right\} = \frac{\bar{P}-\bar{E}}{\bar{\rho}_s} \qquad (27')$$

where the ordinary mean $\overline{\hat{\mathbf{v}}_h}$ of the horizontal velocity vector appears. In the case of an inhomogeneous fluid with horizontal and vertical density gradients the introduction of the weighted mean $\overline{\hat{\hat{\mathbf{v}}}}_h$ which also appears in (26) is useful. These two velocities are connected by $\overline{\hat{\mathbf{v}}_h} = \overline{\hat{\hat{\mathbf{v}}}}_h - \overline{\hat{\mathbf{v}}_h^{**}} = \mathbb{V}_h - \overline{\hat{\mathbf{v}}_h^{**}}$ and therefore the equation of continuity (27') which replaces the general form (27) for incompressible fluids takes the form

$$\frac{\partial(\bar{\eta}+H)}{\partial t} + \nabla_h\cdot\left\{(\bar{\eta}+H)\mathbb{V}_h\right\} = \frac{\bar{P}-\bar{E}}{\bar{\rho}_s} + \nabla_h\cdot\left\{(\bar{\eta}+H)\,\overline{\hat{\mathbf{v}}_h^{**}}\right\} \qquad (27'')$$

where the divergence of the vertically integrated deviations of the velocity vector from its weighted vertical mean acts as a source or sink for the total height $\bar{\eta}+H$ of a vertical column of water.

Writing the general equation of continuity (27) in the following form

$$\frac{\partial(\bar{\eta}+H)}{\partial t} + \nabla_h\cdot\left\{(\bar{\eta}+H)\mathbb{V}_h\right\} = \frac{\bar{P}-\bar{E}}{\overline{\bar{\rho}}} - \frac{\bar{\eta}+H}{\overline{\bar{\rho}}}\left\{\frac{\partial\overline{\bar{\rho}}}{\partial t} + \mathbb{V}_h\cdot\nabla_h\overline{\bar{\rho}}\right\}$$

together with (27") the condition of incompressibility for the vertically averaged density $\overline{\bar{\rho}}$ is

$$\frac{\partial\overline{\bar{\rho}}}{\partial t} + \mathbb{V}_h\cdot\nabla_h\overline{\bar{\rho}} = \frac{1}{\bar{\eta}+H}\left\{\left(1-\frac{\overline{\bar{\rho}}}{\bar{\rho}_s}\right)(\bar{P}-\bar{E}) - \overline{\bar{\rho}}\,\nabla_h\cdot\left\{(\bar{\eta}+H)\,\overline{\hat{\mathbf{v}}_h^{**}}\right\}\right\}$$

replacing $d\rho/dt = 0$ in the non-integrated equations.

The special case of no vertical density variations ($\partial\bar{\rho}/\partial z = 0$; $\overline{\hat{\mathbf{V}}^{**}} = 0$) leads to the simple equation for the mean density

$$\frac{\partial\bar{\bar{\rho}}}{\partial t} + \mathbf{V}_h \cdot \nabla_h \bar{\bar{\rho}} = 0$$

if we assume that $\bar{P} - \bar{E} = 0$.

4.4 Different forms of the vertically integrated equations

In order to get equations whose physical interpretations are easier than (26) we take first the horizontal component of (26) given by

$$\frac{\partial}{\partial t}\left|Q\mathbf{V}_h\right| + \nabla_h \cdot \left|Q\mathbf{V}_h\mathbf{V}_h\right| + (2\boldsymbol{\omega} x Q\mathbf{V})_h$$
$$= gQ\nabla_h\eta^{(T)} - \nabla_h\left|(\bar{\eta}+H)\,\bar{\bar{p}}\right| + (\bar{p}_b - \bar{p}^{(a)})\nabla_h H + \nabla_h\cdot\left\{(\bar{\eta}+H)(-\overline{\rho\mathbf{V}_h''\mathbf{V}_h''} - \overline{\bar{\rho}\hat{\mathbf{V}}_h^{**}\hat{\mathbf{V}}_h^{**}})\right\}$$
$$+ \bar{p}^{(a)}\nabla_h(\bar{\eta}+H) + \overline{\mathbf{N}}_s \cdot (-\overline{\rho\mathbf{V}''\mathbf{V}_h''})^{(a)} + \mathbf{N}_b\cdot(-\overline{\rho\mathbf{V}''\mathbf{V}_h''})_b + (\bar{P} - \bar{E})\,\hat{\mathbf{V}}_{h,s}$$

and separate the pressure into an internal and an external part; i.e. $\bar{\bar{p}} = \bar{\bar{p}}^{(i)} + \bar{p}^{(a)}$. Using then the equation of continuity, (27), we obtain

$$\frac{\partial}{\partial t}\left|Q\mathbf{V}_h\right| + \nabla_h\cdot\left\{Q\mathbf{V}_h\mathbf{V}_h\right\} + (2\boldsymbol{\omega} x Q\mathbf{V})_h$$
$$= Q\left\{\frac{\partial\mathbf{V}_h}{\partial t} + \mathbf{V}_h\cdot\nabla_h\mathbf{V}_h + (2\boldsymbol{\omega} x\mathbf{V})_h + \frac{\bar{P}-\bar{E}}{Q}\mathbf{V}_h\right\}$$
$$= Q\left\{\frac{d_h\mathbf{V}_h}{dt} + (2\boldsymbol{\omega} x\mathbf{V})_h + \frac{\bar{P}-\bar{E}}{Q}\mathbf{V}_h\right\} \tag{28}$$
$$= gQ\nabla_h\eta^{(T)} - \nabla_h\left\{(\bar{\eta}+H)\,\bar{\bar{p}}^{(i)}\right\} + \bar{p}_b^{(i)}\nabla_h H + \nabla_h\cdot\left\{(\bar{\eta}+H)(-\overline{\rho\mathbf{V}_h''\mathbf{V}_h''} - \overline{\bar{\rho}\hat{\mathbf{V}}_h^{**}\hat{\mathbf{V}}_h^{**}})\right\}$$
$$- (\bar{\eta}+H)\nabla_h\bar{p}^{(a)} + \overline{\mathbf{N}}_s\cdot(-\overline{\rho\mathbf{V}''\mathbf{V}_h''})^{(a)} + \mathbf{N}_b\cdot(-\overline{\rho\mathbf{V}''\mathbf{V}_h''})_b + (\bar{P} - \bar{E})\hat{\mathbf{V}}_{h,s}.$$

The vertical component of (26) written as an equation for the unknown internal bottom pressure is given by

$$\overline{p}_b^{(i)} = \overline{p}_b - \overline{p}^{(a)} = Q \left\{ g + \frac{d_h \mathbf{V}_z}{dt} + (2\boldsymbol{\omega} \times \mathbf{V}) \cdot \mathbf{k} + \frac{\overline{P}-\overline{E}}{Q} \left(\mathbf{V}_z - \hat{v}_{z,s} \right) \right.$$

$$- \frac{1}{Q} \nabla_h \cdot \left| (\overline{\eta}+H) (\overline{\overline{-\rho \mathbf{V}_h'' \, V_z''}} - \overline{\overline{\hat{\rho} \hat{\mathbf{V}}_h^{**} \hat{V}_z^{**}}}) \right|$$

$$\left. - \overline{\mathbf{N}}_s \cdot \frac{\overline{(-\rho \mathbf{V}'' \, V_z'')}^{(a)}}{Q} - \mathbf{N}_b \cdot \frac{\overline{(-\rho \mathbf{V}'' \, V_z'')}_b}{Q} \right\} \tag{29}$$

$$= Q \left| g + \mathcal{E}_b \right|$$

where $Q\mathcal{E}_b$ denotes the deviation from the hydrostatically calculated bottom pressure.

Equation (29) expresses the unknown bottom pressure $\overline{p}_b$ by an also unknown function $\mathcal{E}_b$ but this function can be supposed to be small compared to the acceleration g of gravity.

Regarding the vertically integrated internal pressure $(\overline{\eta}+H)\,\overline{\overline{p}}^{(i)}$ a separate calculation using the vertical component of equation (10) is useful if one wishes to introduce deviations from hydrostatic equilibrium.

Starting with the vertical component of (10); i.e.

$$\overline{\rho} \frac{d\hat{V}_z}{dt} = - g \overline{\rho} - \frac{\partial \overline{p}}{\partial z} + \nabla \cdot \left| - \overline{\rho \mathbf{V}'' \, V_z''} \right| - 2\boldsymbol{\omega} \times \overline{\rho} \hat{\mathbf{V}} \cdot \mathbf{k}$$

and integrating twice over depth, first from a depth z to the surface and then from the bottom to the surface one obtains

$$(\overline{\eta}+H)\,\overline{\overline{p}}^{(i)} = (\overline{\eta}+H)(\overline{\overline{p}} - \overline{p}^{(a)}) = g \int_{-H}^{\overline{\eta}} dz \int_{z}^{\overline{\eta}} \overline{\rho} d\zeta + \int_{-H}^{\overline{\eta}} dz \int_{z}^{\overline{\eta}} \overline{\rho} \frac{d\hat{V}_z}{dt} d\zeta$$

$$- \int_{-H}^{\overline{\eta}} dz \int_{z}^{\overline{\eta}} \nabla \cdot (- \overline{\rho \mathbf{V}'' \, V_z''}) \, d\zeta + 2\boldsymbol{\omega} \times \left(\int_{-H}^{\overline{\eta}} dz \int_{z}^{\overline{\eta}} \overline{\rho} \hat{\mathbf{V}} d\zeta \right) \cdot \mathbf{k}. \tag{30}$$

Introduction of the deviation $\overline{\rho}^{*}$ of the density from the vertical mean $\overline{\overline{\rho}}$ and carrying out the integration in the first term on the right hand side gives

$$(\bar{\eta}+H)\,\overline{\overline{p}}^{(i)} = \frac{g}{2\,\overline{\overline{\rho}}}\,Q^2 + g\int_{-H}^{\bar{\eta}} dz \int_{z}^{\bar{\eta}} \bar{\rho}^{\,*}\, d\zeta + \mathcal{E}^{(i)} \tag{31}$$

where the first term gives the hydrostatic value resulting from the mean density; the second gives the effects of the deviations, in the vertical of the density from its vertical mean value, or the hydrostatic value (this is not not necessarily continuous); and the third term $\mathcal{E}^{(i)}$ denotes the deviation of the integrated pressure $(\bar{\eta}+H)\,\overline{\overline{p}}^{(i)}$ from the hydrostatic value.

Assuming hydrostatic equilibrium and disregarding vertical density variations ($\mathcal{E}^{(i)}$, $\bar{\rho}^{*}$ = 0, 0) the form

$$(\bar{\eta}+H)\,\overline{\overline{p}}^{(i)} = \frac{g}{2\,\overline{\overline{\rho}}}\,Q^2 \tag{32}$$

is the well-known "adiabatic law" with the fixed adiabatic exponent 2 used in the so called "hydraulic analogy" expressing that the integrated internal pressure is proportional to the square of the mass of a vertical column of water (Courant and Friedrichs [5] and Tepper [33]).

Introducing the gradient of (31), i.e.

$$\begin{aligned} -\nabla_h\left\{(\bar{\eta}+H)\,\overline{\overline{p}}^{(i)}\right\} &= -g(\bar{\eta}+H)\nabla_h Q - \frac{g}{2}\,Q^2\nabla_h\left(\frac{1}{\overline{\overline{\rho}}}\right) - g\nabla_h\left(\int_{-H}^{\bar{\eta}} dz \int_{z}^{\bar{\eta}} \bar{\rho}^{\,*} d\zeta\right) - \nabla_h\,\mathcal{E}^{(i)} \\ &= -gQ\nabla_h(\bar{\eta}+H) - \frac{g}{2}\,(\bar{\eta}+H)^2\nabla_h\overline{\overline{\rho}} - g\nabla_h\left(\int_{-H}^{\bar{\eta}} dz \int_{z}^{\bar{\eta}} \bar{\rho}^{\,*} d\zeta\right) - \nabla_h\mathcal{E}^{(i)} \end{aligned} \tag{33}$$

and also the following expression for the integrated gradient of the atmospheric pressure

$$-\,(\bar{\eta}+H)\nabla_h\bar{p}^{\,(a)} = gQ\nabla_h\eta^{(a)} + \bar{p}^{\,(a)}Q\,\nabla_h\left(\frac{1}{\overline{\overline{\rho}}}\right) \tag{34}$$

where

$$\eta^{(a)} = -\,\frac{\bar{p}^{\,(a)}}{g\,\overline{\overline{\rho}}} \tag{35}$$

denotes the pressure deficit from some standard pressure, expressed in feet of water, together with (29) one gets from (28) the following set of equations

$$\frac{\partial Q\mathbf{V}_h}{\partial t} + \nabla_h \cdot \left| Q\mathbf{V}_h\mathbf{V}_h \right| + (2\boldsymbol{\omega} x Q\mathbf{V})_h$$

$$= Q\left\{ \frac{d_h\mathbf{V}_h}{dt} + (2\boldsymbol{\omega} x \mathbf{V})_h + \frac{\overline{P}-\overline{E}}{Q}\mathbf{V}_h \right\}$$

$$= -gQ\nabla_h(\overline{\eta}-\eta^{(T)}) + \nabla_h \cdot \left\{ (\overline{\eta}+H)\ (\overline{\overline{-\rho \mathbf{V}_h''\mathbf{V}_h''}} - \overline{\rho}\,\overline{\hat{\mathbf{V}}_h^{**}\hat{\mathbf{V}}_h^{**}}) \right\}$$

$$- \frac{g}{2}(\overline{\eta}+H)^2 \nabla_h \overline{\overline{\rho}} - g\nabla_h \left(\int_{-H}^{\overline{\eta}} dz \int_z^{\overline{\eta}} \overline{\rho}^* d\zeta \right) + \mathcal{E}_b Q \nabla_h H - \nabla_h \mathcal{E}^{(i)}$$

$$- (\overline{\eta}+H)\nabla_h \overline{p}^{(a)} + \overline{\mathbf{N}}_s \cdot (\overline{-\rho \mathbf{V}''\mathbf{V}_h''})^{(a)} + \mathbf{N}_b \cdot (\overline{-\rho \mathbf{V}''\mathbf{V}_h''})_b + (\overline{P}-\overline{E})\hat{\mathbf{V}}_{h,s}. \qquad (36)$$

$$= -gQ\nabla_h\ (\overline{\eta}-\eta^{(T)} - \eta^{(a)}) + \nabla_h \cdot \left| (\overline{\eta}+H)(\overline{\overline{-\rho \mathbf{V}_h''\mathbf{V}_h''}} - \overline{\rho}\,\overline{\hat{\mathbf{V}}_h^{**}\hat{\mathbf{V}}_h^{**}}) \right|$$

$$+ Q\left\{ \frac{g}{2} Q + \overline{p}^{(a)} \right\} \nabla_h\left(\frac{1}{\overline{\overline{\rho}}}\right) - g\nabla_h \left(\int_{-H}^{\overline{\eta}} dz \int_z^{\overline{\eta}} \overline{\rho}^* d\zeta \right) + \mathcal{E}_b Q \nabla_h H - \nabla_h \mathcal{E}^{(i)}$$

$$+ \overline{\mathbf{N}}_s \cdot (\overline{-\rho \mathbf{V}''\mathbf{V}_h''})^{(a)} + \mathbf{N}_b \cdot (\overline{-\rho \mathbf{V}''\mathbf{V}_h''})_b + (\overline{P}-\overline{E})\hat{\mathbf{V}}_{h,s} \qquad (37)$$

which can be used in different combinations for convenience. Together with (27), the equation of continuity, this system is considered always as a system for $\mathbf{V}_h$ and $\overline{\eta}$ whereas $\overline{\overline{\rho}}$, $\overline{\rho}^*$, $\mathcal{E}_b$, $\mathcal{E}^{(i)}$, $(\overline{-\rho \mathbf{V}''\mathbf{V}_h''})_b$, $\hat{\mathbf{V}}_{h,s}$ must be given.

Regarding the deviations $\mathcal{E}^{(i)}$ and $\mathcal{E}_b$ from hydrostatic equilibrium, however, a further approach can be given. Considering first the second term on the right hand side in equation (30) and introducing here also the deviations $\overline{\rho}^*$ from the vertical mean of the density one obtains approximately

$$\int_{-H}^{\overline{\eta}} dz \int_z^{\overline{\eta}} \overline{\rho} \frac{d\hat{V}_z}{dt} d\zeta = \overline{\overline{\rho}} \int_{-H}^{\overline{\eta}} dz \int_z^{\overline{\eta}} \frac{d\hat{V}_z}{dt} d\zeta + \int_{-H}^{\overline{\eta}} dz \int_z^{\overline{\eta}} \overline{\rho}^* \frac{d\hat{V}_z}{dt} d\zeta$$

$$\approx \overline{\overline{\rho}} \int_{-H}^{\overline{\eta}} dz \int_z^{\overline{\eta}} d\left(\frac{\hat{V}_z^2}{2}\right) + \int_{-H}^{\overline{\eta}} dz \int_z^{\overline{\eta}} \overline{\rho}^* \frac{d\hat{V}_z}{dt} d\zeta$$

$$\approx Q\left(\frac{\hat{V}_z^2}{2}\right)_s - Q\left(\frac{\overline{\overline{\hat{V}_z^2}}}{2}\right) + \int_{-H}^{\overline{\eta}} dz \int_z^{\overline{\eta}} \overline{\rho}^* \frac{d\hat{V}_z}{dt} d\zeta$$

We therefore get for $\varepsilon^{(i)}$ the expression

$$\varepsilon^{(i)} = Q\left\{\left(\frac{\hat{V}_z^2}{2}\right)_s - \left(\overline{\frac{\hat{V}_z^2}{2}}\right)\right\} + \int_{-H}^{\bar{\eta}} dz \int_{z}^{\bar{\eta}} \bar{\rho}^* \frac{d\hat{V}_z}{dt}\, d\zeta - \int_{-H}^{\bar{\eta}} dz \int_{z}^{\bar{\eta}} \nabla \cdot \left\{\overline{-\rho \mathbf{V}'' V_z''}\right\} dz$$

$$+ 2\boldsymbol{\omega} \times \left(\int_{-H}^{\bar{\eta}} dz \int_{z}^{\bar{\eta}} \bar{\rho} \hat{\mathbf{V}} d\zeta\right) \cdot \mathbf{k} \qquad (38)$$

where the term containing $(\hat{V}_z)_s$ can be expressed further by $\bar{\eta}$ with the aid of of the kinematic boundary condition (13). It appears likely that the first term on the right hand side, i.e. $Q\left\{\left(\frac{\hat{V}_z^2}{2}\right)_s - \left(\overline{\frac{\hat{V}_z^2}{2}}\right)\right\}$ gives a good approximation for $\varepsilon^{(i)}$ in practical cases.

The same approach can be found for ε_b by integrating the vertical component of (10) over the total depth leading to the expression for ε_b in the form

$$\varepsilon_b = \frac{1}{\bar{\eta}+H}\left\{\left(\frac{\hat{V}_z^2}{2}\right)_s - \left(\frac{\hat{V}_z^2}{2}\right)_b\right\} + \overline{\bar{\rho}^* \frac{d\hat{V}_z}{dt}} - \overline{\nabla \cdot \left[\overline{-\rho \mathbf{V}'' V_z''}\right]} + 2\boldsymbol{\omega} \times Q\mathbb{V} \cdot \mathbf{k} \qquad (39)$$

where again the kinematic boundary conditions (13), (14) can be introduced in order to eliminate $(\hat{V}_z)_s$ and $(\hat{V}_z)_b$. It appears that a sufficiently good approach for many cases is given by

$$\varepsilon_b \approx \frac{1}{\bar{\eta}+H}\left(\frac{\hat{V}_z^2}{2}\right)_s \approx \frac{1}{\bar{\eta}+H}\left(\frac{1}{2}\left(\frac{\partial \bar{\eta}}{\partial t}\right)^2\right).$$

In concluding this section, we consider the case in which viscous and small-scale turbulent friction can be neglected and in which all typical hurricane effects are disregarded; i.e., $(\bar{p}^{(a)}, (\overline{-\rho \mathbf{V}'' \mathbf{V}_h''})^{(a)}, \bar{P}-\bar{E}) = (0,0,0)$, and $\bar{\rho}$ = const. Neglecting further the astronomical tidal potential gradient, equation (28) can be written

$$Q\left\{\frac{d_h \mathbb{V}_h}{dt} + 2\boldsymbol{\omega} \times \mathbb{V}_h\right\} = -\nabla_h \left\{(\bar{\eta}+H)\, \overline{\bar{p}^{(i)}}\right\} + gQ\nabla_h H$$

$$+ \varepsilon_b Q \nabla_h H + \nabla_h \cdot \left\{(\bar{\eta}+H)\left(\overline{-\bar{\rho}\hat{\mathbf{V}}_h^{**}\hat{\mathbf{V}}_h^{**}}\right)\right\}$$

The first approximation of the shallow-water wave theory, described by

$$Q\left\{\frac{d_h \mathbb{V}_h}{dt} + 2\boldsymbol{\omega}\times\mathbb{V}_h\right\} = -\nabla_h\left\{(\bar{\eta}+H)\,\overline{\overline{p}}^{(i)}\right\} + gQ\nabla_h H$$

$$(\bar{\eta}+H)\,\overline{\overline{p}}^{(i)} = \frac{g}{2\,\overline{\overline{\rho}}}\,Q^2 \tag{40}$$

$$\frac{\partial Q}{\partial t} + \nabla_h\cdot\left\{Q\mathbb{V}_h\right\} = 0$$

is obtained by neglecting the terms

$$\varepsilon_b Q\nabla_h H, \quad \nabla_h\cdot\left\{\overline{\overline{(\bar{\eta}+H)(-\bar{\rho}\hat{\mathbf{V}}_h^{**}\hat{\mathbf{V}}_h^{**})}}\right\}, \quad \mathcal{E}^{(i)}.$$

Therefore these terms represent the higher approximations of this theory.

Neglecting these terms also in the general equations (28), (36) or (37) means that the resulting equations are some kind of first approximation of the general equations and show the same restrictions that are included in the first approximation of the shallow-water wave theory (Stoker [29], Wehausen and Laitone [38]). This will be considered later after having performed the second time averaging.

5. STORM SURGE EQUATIONS

5.1 Second time averaging

We use a second process of time averaging in order to obtain equations for wave periods on the order of storm surge periods and longer. The time interval may be of the order of a few minutes.

Using the same definitions for ordinary and weighted time mean as given in 2.1 the averaged equation of continuity (27) is then given by

$$\frac{\partial\bar{Q}}{\partial t} + \nabla_h\cdot\left\{\bar{Q}\hat{\mathbb{V}}_h\right\} = \frac{\hat{d}_h\bar{Q}}{dt} + \bar{Q}\nabla_h\cdot\hat{\mathbb{V}}_h = \overline{\overline{P}}-\overline{\overline{E}} \tag{41}$$

whereas the averaged equations (28), (36) and (37) are

$$\frac{\partial}{\partial t}\left\{\bar{Q}\hat{\mathbb{V}}_h\right\} + \nabla_h\cdot\left\{\bar{Q}\hat{\mathbb{V}}_h\hat{\mathbb{V}}_h\right\} + (2\boldsymbol{\omega}\times\bar{Q}\hat{\mathbb{V}})_h = \bar{Q}\left\{\frac{\hat{d}_h\hat{\mathbb{V}}_h}{dt} + (2\boldsymbol{\omega}\times\mathbb{V})_h + \frac{\overline{\overline{P}}-\overline{\overline{E}}}{\bar{Q}}\mathbb{V}_h\right\}$$

$$= g\bar{Q}\nabla_h\eta^{(T)} - \nabla_h\left\{\overline{(\bar{\eta}+H)\overline{\overline{p}}_b^{(i)}}\right\} + \overline{\overline{p}}_b^{(i)}\nabla_h H + \nabla_h\cdot\left\{\overline{(\bar{\eta}+H)(-\overline{\overline{\rho\mathbf{V}_h''\mathbf{V}_h''}})} + \overline{(\bar{\eta}+H)(-\overline{\overline{\rho\hat{\mathbf{V}}_h^{**}\hat{\mathbf{V}}_h^{**}}})}\right.$$

$$\left. + (-Q\mathbb{V}_h''\mathbb{V}_h'')\right\} - \overline{(\bar{\eta}+H)\nabla_h\bar{p}^{(a)}} + \overline{\mathbf{N}_s\cdot(\overline{-\rho\mathbf{V}''\mathbf{V}_h''})^{(a)}} + \mathbf{N}_b\cdot\overline{\overline{(-\rho\mathbf{V}''\mathbf{V}_h'')_b}} + \overline{(\bar{P}-\bar{E})\hat{\mathbf{V}}_{h,s}}$$

$$= -g\overline{Q\nabla_h\bar{\bar{\eta}}} + g\bar{Q}\nabla_h\eta^{(T)} + \nabla_h\cdot\left\{\overline{(\bar{\eta}+H)(-\rho\overline{\overline{\mathbf{V}_h''\mathbf{V}_h''}})} + \overline{(\bar{\eta}+H)(-\bar{\rho}\overline{\hat{\mathbf{V}}_h^{**}\hat{\mathbf{V}}_h^{**}})} + \overline{(-Q\mathbb{V}_h''\mathbb{V}_h'')}\right\}$$

$$-\frac{g}{2}\overline{(\bar{\eta}+H)^2\nabla_h\bar{\bar{\rho}}} - g\nabla_h\overline{\left(\int_{-H}^{\bar{\eta}}dz\int_z^{\bar{\eta}}\bar{\rho}^*d\zeta\right)} + \mathcal{E}_b Q\nabla_h H - \nabla_h\overline{\mathcal{E}^{(i)}}$$

$$-\overline{(\bar{\eta}+H)\nabla_h\bar{p}^{(a)}} + \overline{\mathbf{N}_s\cdot\overline{(-\rho\mathbf{V}''\mathbf{V}_h'')}^{(a)}} + \mathbf{N}_b\cdot\overline{(-\rho\mathbf{V}''\mathbf{V}_h'')}_b + \overline{(\bar{P}-\bar{E})\hat{\mathbf{V}}_{h,s}}$$

$$= -g\overline{Q\nabla_h\bar{\eta}} + g\bar{\bar{Q}}\nabla_h\eta^{(T)} + g\overline{Q\nabla_h\eta^{(a)}} + \nabla_h\cdot\left\{\overline{(\bar{\eta}+H)(-\rho\overline{\overline{\mathbf{V}_h''\mathbf{V}_h''}})} + \overline{(\bar{\eta}+H)(-\bar{\rho}\overline{\hat{\mathbf{V}}_h^{**}\hat{\mathbf{V}}_h^{**}})}\right.$$

$$\left.+\overline{(-\bar{Q}\mathbb{V}_h''\mathbb{V}_h'')}\right\} + \overline{Q\left\{\frac{g}{2}Q + \bar{p}^{(a)}\right\}\nabla_h\left(\frac{1}{\bar{\bar{\rho}}}\right)} - g\nabla_h\overline{\left(\int_{-H}^{\bar{\eta}}dz\int_z^{\bar{\eta}}\bar{\rho}^*d\zeta\right)} + \mathcal{E}_b Q\nabla_h H - \nabla_h\overline{\mathcal{E}^{(i)}}$$

$$+\overline{\bar{\mathbf{N}}_s\cdot\overline{(-\rho\mathbf{V}''\mathbf{V}_h'')}^{(a)}} + \mathbf{N}_b\cdot\overline{(-\rho\mathbf{V}''\mathbf{V}_h'')}_b + \overline{(\bar{P}-\bar{E})\hat{\mathbf{V}}_{h,s}}$$

Introducing for the time mean of each product term the mean value and the corresponding correlation product one gets this for the second representation

$$\frac{\partial}{\partial t}\left[\bar{Q}\hat{\mathbb{V}}_h\right] + \nabla_h\cdot\left\{\bar{Q}\hat{\mathbb{V}}_h\hat{\mathbb{V}}_h\right\} + (2\omega\times\bar{Q}\hat{\mathbb{V}})_h = \bar{Q}\left\{\frac{\hat{d}_h\hat{\mathbb{V}}_h}{dt} + (2\omega\times\hat{\mathbb{V}})_h + \frac{\bar{\bar{P}}-\bar{\bar{E}}}{\bar{Q}}\hat{\mathbb{V}}_h\right\}$$

$$= -g\bar{Q}\nabla_h(\bar{\bar{\eta}}-\eta^{(T)}) + \nabla_h\cdot\left\{\overline{(\bar{\eta}+H)(-\rho\overline{\overline{\mathbf{V}_h''\mathbf{V}_h''}})} + \overline{(\bar{\eta}+H)(-\bar{\rho}\overline{\hat{\mathbf{V}}_h^{**}\hat{\mathbf{V}}_h^{**}})} + \overline{(-Q\mathbb{V}_h''\mathbb{V}_h'')}\right\}$$

$$-g\bar{\bar{\bar{\rho}}}\nabla_h\left(\frac{\overline{\bar{\eta}'^2}}{2}\right) - \frac{g}{2}\left\{(\bar{\bar{\eta}}+H)^2 + (\overline{\bar{\eta}'^2})\right\}\nabla_h\bar{\bar{\bar{\rho}}} - g\overline{\left(\int_{-H}^{\bar{\eta}}dz\int_z^{\bar{\eta}}\bar{\rho}^*d\zeta\right)} + \bar{\mathcal{E}}_b\bar{Q}\nabla_h H + \overline{\mathcal{E}_b'Q}\nabla_h H - \nabla_h\overline{\mathcal{E}^{(i)}}$$

$$-g\nabla_h\overline{\left(\bar{\bar{\rho}}'\frac{\bar{\eta}'^2}{2}\right)} - g(\bar{\bar{\eta}}+H)\nabla_h\overline{(\bar{\bar{\rho}}'\bar{\eta}')} - (\bar{\bar{\eta}}+H)\nabla_h\bar{\bar{p}}^{(a)} + \bar{\bar{\mathbf{N}}}_s\cdot\overline{\overline{(-\rho\mathbf{V}''\mathbf{V}_h'')}}^{(a)}$$

$$+\mathbf{N}_b\cdot\overline{(-\rho\mathbf{V}''\mathbf{V}_h'')}_b + (\bar{\bar{P}}-\bar{\bar{E}})\hat{\bar{\mathbf{V}}}_{h,s} - \overline{\bar{\eta}'\nabla_h\bar{p}'^{(a)}} + \overline{\bar{\mathbf{N}}_s'\cdot(-\rho\mathbf{V}''\mathbf{V}_h'')'^{(a)}} + \overline{(\bar{P}'-\bar{E}')\hat{\mathbf{V}}_{h,s}'} \tag{42}$$

This equation has been written in such a way that the analogy to two-dimensional compressible flow is apparent. Writing (41) and (42) in abridged form, i.e.

$$\frac{\hat{d}_h\bar{Q}}{dt} + \bar{Q}\nabla_h\cdot\hat{\mathbb{V}}_h = \bar{\bar{P}}-\bar{\bar{E}} \qquad\qquad \frac{\hat{d}_h\hat{\mathbb{V}}_h}{dt} + 2\omega\times\hat{\mathbb{V}}_h = \frac{1}{\bar{Q}}\mathbb{F}_h.$$

the system in fact describes the two-dimensional compressible flow with "density" $\bar{Q}$ (mass per unit area), velocity $\hat{\mathbb{V}}_h$, mass generation $\bar{\bar{P}}-\bar{\bar{E}}$ (mass per unit area and time), and with the sum $\mathbb{F}_h$ of some "forces" (force per unit area).

The physical interpretation of the terms in equation (42) is as follows.

The left hand side contains terms connected with the mass transport of larger scale, i.e. mass transport connected with the surge and the astronomical tidal wave. Here

$$\bar{Q} = \overline{\left(\int_{-H}^{\bar{\eta}} \bar{\rho} dz\right)}, \quad \bar{Q}\hat{\mathbb{V}}_h = \overline{\left(\int_{-H}^{\bar{\eta}} \bar{\rho}\hat{\mathbf{V}}_h dz\right)}, \quad (\bar{\bar{\eta}}+H)\hat{\mathbb{V}}_h \approx \frac{1}{\bar{\bar{\bar{\rho}}}}\overline{\left(\int_{-H}^{\bar{\eta}} \bar{\rho}\hat{\mathbf{V}}_h dz\right)} .$$

are the mass of a vertical column, the mass transport vector and the volume transport vector respectively. The volume transport vector has the physical meaning of a barycentric velocity.

Regarding the Coriolis acceleration there is no doubt that this term can be approximated in the usual way

$$(2\boldsymbol{\omega}\times\hat{\mathbb{V}})_h \approx f\mathbf{k}\times\hat{\mathbb{V}}_h, \qquad f = 2\,\omega \sin\phi \tag{43}$$

The right hand terms are acting as "forces" to produce the mean horizontal mass transport $\bar{Q}\hat{\mathbb{V}}_h$ within the surge. These are:

(1) <u>The well-known force resulting from the mean inclination</u> $\bar{\bar{\eta}}$ <u>of the surface</u>

$$-\, g\,\bar{Q}\,\nabla_h\,\bar{\bar{\eta}} \tag{44}$$

(2) <u>Generalized lateral friction</u>

$$\nabla_h\cdot\bar{\mathcal{F}}_h = \nabla_h\cdot\left\{\overline{(\bar{\eta}+H)(\overline{\overline{-\bar{\rho}\mathbf{V}_h''\mathbf{V}_h''}})} + \overline{(\bar{\eta}+H)(\overline{\overline{-\bar{\rho}\hat{\mathbf{V}}_h^{**}\hat{\mathbf{V}}_h^{**}}})} + \overline{(-Q\mathbb{V}_h''\mathbb{V}_h'')}\right\} \tag{45}$$

where the lateral "friction"stress tensor consists of three different turbulent friction tensors resulting from three different averages of the nonlinear terms of the equations of motion.

Following the usual assumptions regarding the correlation products (45), it can be written formally (Eliassen and Kleinschmidt [9])

$$\begin{aligned}
\overline{-\rho\mathbf{V}_h''\mathbf{V}_h''} &= \bar{\rho}\,K_h^{(1)}\,\nabla_h\hat{\mathbf{V}}_h \\
\overline{\overline{-\bar{\rho}\hat{\mathbf{V}}_h^{**}\hat{\mathbf{V}}_h^{**}}} &= \bar{\bar{\rho}}\,K_h^{(2)}\,\nabla_h\mathbb{V}_h \quad ; \quad \mathbb{V}_h = \hat{\hat{\mathbf{V}}}_h \\
\overline{-Q\mathbb{V}_h''\mathbb{V}_h''} &= \bar{Q}\,K_h^{(3)}\,\nabla_h\hat{\mathbb{V}}_h
\end{aligned} \tag{46}$$

where the coefficients $\overset{(1)}{K}_h$, $\overset{(2)}{K}_h$ and $\overset{(3)}{K}_h$ of eddy viscosity certainly have different meanings and may be expected to have different values. The first one is connected with small-scale turbulent motion averaged over a time interval of the order of seconds. The second one describes the structure of the velocity distribution in the vertical and the third one describes turbulent lateral friction connected with the mean motion within the surge averaged over a time interval of a few minutes.

Introducing (46) into (45) gives formally

$$\nabla_h \cdot \overline{\mathcal{F}}_h = \nabla_h \cdot \left\{ \overline{(\bar{\eta}+H)\, \bar{\rho}\, \overset{(1)}{K}_h \nabla_h \hat{\mathbf{v}}_h} + \overline{(\bar{\eta}+H)\, \bar{\rho}\, \overset{(2)}{K}_h \nabla_h \overline{\mathbb{V}}_h} + \bar{Q}\, \overset{(3)}{K}_h \nabla_h \hat{\overline{\mathbb{V}}}_h \right\}$$

$$= \nabla_h \cdot \left\{ \widehat{(\bar{\eta}+H)\, \bar{\rho}\, \overset{(1)}{K}_h \nabla_h \hat{\mathbf{v}}_h} + \bar{Q}\, \widehat{\overset{(2)}{K}_h \nabla_h \overline{\mathbb{V}}_h} + \bar{Q}\, \overset{(3)}{K}_h \nabla_h \hat{\overline{\mathbb{V}}}_h \right\} \tag{47}$$

$$= \nabla_h \cdot \left\{ \bar{Q} \left(\widehat{\overset{(1)}{K}_h \nabla_h \hat{\mathbf{v}}_h} + \widehat{\overset{(2)}{K}_h \nabla_h \overline{\mathbb{V}}_h} + \overset{(3)}{K}_h \nabla_h \hat{\overline{\mathbb{V}}}_h \right) \right\}$$

which might be written but only with rough approximation, as

$$\nabla_h \cdot \overline{\mathcal{F}}_h \approx \nabla_h \cdot \left\{ \bar{Q}\, K_h \nabla_h \hat{\overline{\mathbb{V}}}_h \right\} \approx K_h \nabla_h^2 \left(\bar{Q} \hat{\mathbb{V}}_h \right) \tag{48}$$

where very little is known about the new coefficient of eddy viscosity K_h.

(3) Boundary friction stresses:

(a) At the surface we have

$$\overline{\overline{\mathbf{N}_s \cdot (\overline{-\rho \mathbf{v}'' \mathbf{v}_h''})^{(a)}}} = \overline{\overline{\mathbf{N}}}_s \cdot (\overline{-\rho \mathbf{v}'' \mathbf{v}_h''})^{(a)} + \overline{\overline{\mathbf{N}_s' \cdot (-\rho \mathbf{v}'' \mathbf{v}_h'')'^{(a)}}}$$

$$= (\overline{\overline{-\rho \mathbf{v}_z'' \mathbf{v}_h''}})^{(a)} - \nabla_h \bar{\bar{\eta}} \cdot (\overline{\overline{-\rho \mathbf{v}_h'' \mathbf{v}_h''}})^{(a)} - \overline{\nabla_h \bar{\eta}' \cdot (\overline{-\rho \mathbf{v}_h'' \mathbf{v}_h''})'^{(a)}}$$

$$\overline{\overline{\mathbf{N}_s \cdot (\overline{-\rho \mathbf{v}'' \mathbf{v}_h''})^{(a)}}} = \overline{\mathcal{T}}_{z,h}^{(a)} - \nabla_h \bar{\bar{\eta}} \cdot \left(\mathbf{i} \overline{\mathcal{T}}_{x,h}^{(a)} + \mathbf{j} \overline{\mathcal{T}}_{y,h}^{(a)} \right) - \overline{\nabla_h \bar{\eta}' \cdot \left(\mathbf{i} \mathcal{T}_{x,h}'^{(a)} + \mathbf{j} \mathcal{T}_{y,h}'^{(a)} \right)} \tag{49}$$

where the definitions $\overline{\mathbf{N}}_s = \mathbf{k} - \nabla_h \bar{\eta}$ from 3.1 and $\mathcal{T}$ from (4) have been used.

(b) At the bottom

$$\mathbf{N}_b \cdot (\overline{\overline{-\rho \mathbf{v}'' \mathbf{v}_h''}})_b = -(\overline{\overline{-\rho \mathbf{v}_z'' \mathbf{v}_h''}})_b - \nabla_h H \cdot (\overline{\overline{-\rho \mathbf{v}_h'' \mathbf{v}_h''}})_b$$

$$\mathbf{N}_b \cdot (\overline{\overline{-\rho \mathbf{v}'' \mathbf{v}_h''}})_b = -(\overline{\mathcal{T}}_{z,h})_b - \nabla_h H \cdot \left\{ \mathbf{i} (\overline{\mathcal{T}}_{x,h})_b + \mathbf{j} (\overline{\mathcal{T}}_{y,h})_b \right\} \tag{50}$$

with $\mathbf{N}_b = -(\mathbf{k}+\nabla_h H)$. This expression is shorter than (49) because it can be assumed in this problem that the bottom is not a random function of time. The expressions

$$\overline{\overline{\mathbf{N}}_s \cdot (\overline{-\rho\mathbf{V}''\mathbf{V}_h''})^{(a)}} \approx \overline{\mathfrak{T}}^{(a)}_{z,h} \tag{51}$$

$$\mathbf{N}_b \cdot (\overline{\overline{-\rho\mathbf{V}''\mathbf{V}_h''}})_b \approx -(\overline{\mathfrak{T}}_{z,h})_b \tag{52}$$

are widely used. Strictly speaking, these are valid only for a flat bottom and only for a very long wave with a gentle slope and without any superimposed surface waves of finite height and shorter period.

If a complete spectrum of waves is present and if the surface wind field is turbulent, equation (51) does not provide sufficient information about the effect of surface friction from a theoretical point of view. A better approximation is provided by

$$\overline{\overline{\mathbf{N}}_s \cdot (\overline{-\rho\mathbf{V}''\mathbf{V}_h''})^{(a)}} \approx \overline{\mathfrak{T}}^{(a)}_{z,h} - \overline{\nabla_h\overline{\eta}' \cdot (\mathbf{i}\,\mathfrak{T}'^{(a)}_{x,h} + \mathbf{j}\,\mathfrak{T}'^{(a)}_{y,h})}. \tag{53}$$

More empirical data are needed for a full evaluation of the corrective terms.

(4) Atmospheric pressure gradient

The so called inverted barometer effect is connected with the external atmospheric pressure gradient in the case of equilibrium and is given here by

$$-\overline{(\overline{\eta}+H)\nabla_h\,\overline{p}^{(a)}} = -(\overline{\overline{\eta}}+H)\nabla_h\,\overline{\overline{p}}^{(a)} - \overline{\overline{\eta}'\,\nabla_h\,p'^{(a)}}$$

where an additional term connected with small-scale variations of the pressure gradient appears. This is small even under hurricane conditions.

(5) Wave set-up term

The wave set-up term in (42), i.e.

$$-g\,\overline{\overline{\overline{\rho}}}\,\nabla_h\left(\frac{\overline{\overline{\eta}'^2}}{2}\right)$$

is a typical non-linear effect of the equations of motion. All superimposed waves contribute to this term. The effect is proportional to the negative horizontal gradient of the variance $(\overline{\overline{\eta}'^2})$ of the superimposed waves which usually has a maximum in the breaker zone near the coast. If no other effects are present the mean surface elevation $\overline{\overline{\eta}}$ should have a minimum here and may increase sharply between the breaker zone and the beach. In the open sea this effect is small; however, near the coast under hurricane conditions this effect can be very important.

(6) Horizontal density gradients and vertical density variations

Horizontal density gradients build up a force in the equation for horizontal mass transport (42), i.e.

$$- \frac{g}{2} \left\{ (\bar{\bar{\eta}}+H)^2 + (\overline{\bar{\eta}'^2}) \right\} \nabla_h \bar{\bar{\rho}}$$

which is directed toward smaller mean density values $\bar{\bar{\rho}}$ (Hansen [14]). In the case of equilibrium, the mean surface elevation $\bar{\bar{\eta}}$ should have a maximum where the mean density has a minimum, as is the case along the axis of the Gulf Stream near the coast of Florida. Such density gradients affect the storm surge by resonance under certain conditions.

The influence of the vertically integrated deviation $\bar{\rho}^*$ from the vertical mean $\bar{\bar{\rho}}$ is given by

$$- g\nabla_h \overline{\left(\int_{-H}^{\bar{\eta}} dz \int_{z}^{\bar{\eta}} \bar{\rho}^* d\zeta \right)},$$

where discontinuities of density in the vertical are allowed even with non-horizontal boundary surfaces between the various layers. These boundary surfaces, however, have to be known in each theory using the vertically integrated equations of motion integrating over the whole depth.

(7) The terms connected with density variations with time

The terms

$$- \frac{g}{2} \nabla_h (\overline{\bar{\rho}'\bar{\eta}'^2}) - g (\bar{\bar{\eta}}+H) \nabla_h (\overline{\bar{\rho}'\bar{\eta}'})$$

represent correlations between local variations of the vertically averaged density with time and the small surface waves which are surperimposed on the surge. Such density variations can be caused by rain or suspended sediment even in incompressible fluids. They can be important at the mean actual coast where at a fixed point during the time interval of the second averaging the water is oscillating around the mean actual coast line which varies with time more slowly than the instantaneous water line. Except for this possibility of importance near the coast, these terms can be neglected.

(8) Tidal generating force

The term containing the gradient of the astronomical tidal potential

$$g \bar{Q}\nabla_h \eta^{(T)} = - \bar{Q} \nabla_h \Phi^{(T)}$$

is small for scales compared with hurricane dimensions. Interactions between the tidal wave current and a basic current can be treated by subtracting these currents from the basic equation (42) leaving then an equation only for the hurricane storm surge.

(9) Non-hydrostatic effects

These effects are contained in the two terms $\overline{\mathcal{E}_b Q}\nabla_h H$ and $-\nabla_h \overline{\mathcal{E}^{(i)}}$.

Introducing the expressions (39), (38) for $\mathcal{E}_b$ and $\mathcal{E}^{(i)}$ one gets explicitly

$$
\begin{aligned}
\overline{\mathcal{E}_b Q}\nabla_h H - \nabla_h \overline{\mathcal{E}^{(i)}} \approx & \left[\overline{\overline{\overline{\rho}}\left\{ \left(\frac{\hat{V}_z^2}{2}\right)_s - \left(\frac{\hat{V}_z^2}{2}\right)_b \right\}} + \overline{\overline{Q\, \overline{\rho}^* \frac{d\hat{V}_z}{dt}}} + \ldots \right] \nabla_h H \\
& - \nabla_h \left[\overline{Q \left\{ \left(\frac{\hat{V}_z^2}{2}\right)_s - \overline{\overline{\left(\frac{\hat{V}_z^2}{2}\right)}} \right\}} + \overline{\left(\int_{-H}^{\overline{\eta}} dz \int_z^{\overline{\eta}} \overline{\rho}^* \frac{d\hat{V}_z}{dt} d\zeta \right)} + \ldots \right] \\
= & \left[\overline{\overline{\overline{\rho}}} \left\{ \overline{\left(\frac{\hat{V}_z^2}{2}\right)_s} - \overline{\left(\frac{\hat{V}_z^2}{2}\right)_b} \right\} + \overline{\overline{\overline{\rho}} \left\{ \left(\frac{\hat{V}_z^2}{2}\right)_s' - \left(\frac{\hat{V}_z^2}{2}\right)_b' \right\}} + \overline{Q\, \overline{\rho}^* \overline{\overline{\frac{d\hat{V}_z}{dt}}}} + \ldots \right] \nabla_h H \\
& - \nabla_h \left[\overline{Q} \left\{ \overline{\left(\frac{\hat{V}_z^2}{2}\right)_s} - \overline{\overline{\left(\frac{\hat{V}_z^2}{2}\right)}} \right\} + \overline{Q' \left\{ \left(\frac{\hat{V}_z^2}{2}\right)_s' - \overline{\overline{\left(\frac{\hat{V}_z^2}{2}\right)'}} \right\}} + \ldots \right]
\end{aligned} \qquad (54)
$$

where the friction and Coriolis terms have not been written. The boundaries can be expressed in terms of $\overline{\eta}$ and H by introducing the kinematic conditions (13) and (14). After the second time averaging process has been carried out, the non-hydrostatic terms can be expressed in terms of the mean motion of the surface and correlation between the elevations $\overline{\eta}'$ of the superimposed waves and other quantities.

The non-hydrostatic terms represent one force in the direction of $\nabla_h H = -\nabla_h(-H)$ and one in the direction of $-\nabla_h \overline{\mathcal{E}^{(i)}}$. The first one vanishes for a flat bottom. In this case the second one is small. However, if the bottom slope is great, the induced vertical velocities are also greater and therefore the non-hydrostatic effects become significant.

Writing these terms in the following form

$$
\overline{\mathcal{E}_b Q}\,\nabla_h H - \nabla_h \overline{\mathcal{E}^{(i)}} = \overline{\mathcal{E}}_b \overline{Q} \nabla_h H + \left(\overline{\overline{\overline{\rho}}}\; \overline{\overline{\eta}' \mathcal{E}_b'} + (\overline{\overline{\eta}}+H)\, \overline{\overline{\overline{\rho}}' \mathcal{E}_b'} + \overline{\overline{\overline{\rho}}' \overline{\eta}' \mathcal{E}_b'} \right) \nabla_h H - \nabla_h \overline{\mathcal{E}}^{(i)}
$$

a correlation between the turbulent pressure variations at the bottom (non-hydrostatic) and the amplitudes of the superimposed surface waves appears which may be important in connection with special bottom configurations near the coast.

(10) Precipitation-minus-evaporation effect

In the case of mass exchange at the surface of the sea the complete kinematic boundary condition (15) led to a source term in the equation of continuity and also to a source for the mass transport in the equation of motion given by

$$\overline{(\overline{P} - \overline{E})\,\hat{\mathbf{V}}_{h,s}} = (\overline{\overline{P}} - \overline{\overline{E}})\,\overline{\hat{\mathbf{V}}_{h,s}} + \overline{(\overline{P}' - \overline{E}')\,\hat{\mathbf{V}}'_{h,s}}\ .$$

Here either the instantaneous horizontal surface velocity vector $\hat{\mathbf{V}}_{h,s}$ has to be known or its decomposition into the time mean and the turbulent fluctuations. This effect is small even if there is heavy rain within the hurricane and can be neglected in the open sea and in not too shallow water. However, in small basins with shallow water this effect may be important.

5.2 Basic equations for long-period motions of the ocean

The equations derived so far are quite general and the introduction of approximations or special assumptions has been largely avoided in the derivation of the system (41), (42) for the mass transport within the ocean under the most general conditions of an atmospheric wind- pressure- and rain-field acting on the surface of the sea. The effects of surface run-off of rainwater falling on the land bordering the sea and of rivers emptying into the sea have been neglected.

Not even the conditions of incompressibility or homogeneity have been used and therefore this system can be applied to the description of many atmospheric motions as well as to oceanic motions if the boundary conditions are suitably modified. If the vertical averaging is applied to the total mass of the atmosphere, the two-boundary value problem of oceanography is simplified to a one-boundary value problem for the atmosphere, but in this case the turbulent nature of the sea surface as expressed in equation (49) may need to be included in the bottom boundary condition (50). The application of this technique to atmospheric motions will not be carried out here but the reader is referred to Van Miegham [36] for a discussion.

Writing down the system (41), (42) once more using the approximation (43) for the Coriolis acceleration and the notations (45), (49), and (50) the final system is given by

$$\frac{\partial \bar{Q}}{\partial t} + \nabla_h \cdot \left\{ \bar{Q} \hat{\mathbb{V}}_h \right\} = \bar{\bar{P}} - \bar{\bar{E}} \tag{41}$$

$$\frac{\partial \bar{Q} \hat{\mathbb{V}}_h}{\partial t} + \nabla_h \cdot \left\{ \bar{Q} \hat{\mathbb{V}}_h \hat{\mathbb{V}}_h \right\} + f\mathbf{k} \times \bar{Q} \hat{\mathbb{V}}_h - \nabla_h \cdot \overline{\mathcal{F}}_h$$

$$= -g\bar{Q}\nabla_h \bar{\bar{\eta}} \quad - g\,\bar{\bar{\bar{\rho}}}\,\nabla_h\left(\frac{\overline{\bar{\eta}'^2}}{2}\right) \quad - g(\bar{\bar{\eta}}+H)\nabla_h\left(\overline{\bar{\bar{\rho}}'\bar{\eta}'}\right) -$$

$$- g\nabla_h\left(\overline{\bar{\bar{\rho}}'\bar{\eta}'^2}/2\right) +$$

$$+ g\bar{Q}\nabla_h \eta^{(T)} \quad -$$

$$- \frac{g}{2}\left\{(\bar{\bar{\eta}}+H)^2 + (\overline{\bar{\eta}'^2})\right\}\nabla_h \bar{\bar{\bar{\rho}}} -$$

$$- g\nabla_h\left(\overline{\int_{-H}^{\bar{\eta}} dz \int_{z}^{\bar{\eta}} \bar{\rho}^* d\zeta}\right) -$$

$$-(\bar{\bar{\eta}}+H)\nabla_h \bar{\bar{p}}^{(a)} \qquad - \overline{\bar{\eta}'\nabla_h \bar{p}'^{(a)}} + \tag{55}$$

$$+ \bar{\mathcal{T}}^{(a)}_{z,h} - \overline{\nabla_h \bar{\eta}' \cdot (\mathbf{i}\mathcal{T}'^{(a)}_{x,h} + \mathbf{j}\mathcal{T}'^{(a)}_{y,h})} - \nabla_h \bar{\bar{\eta}} \cdot (\mathbf{i}\bar{\mathcal{T}}^{(a)}_{x,h} + \mathbf{j}\bar{\mathcal{T}}^{(a)}_{y,h}) -$$

$$- (\bar{\mathcal{T}}_{z,h})_b \qquad - \nabla_h H \cdot \left\{\mathbf{i}(\bar{\mathcal{T}}_{x,h})_b + \mathbf{j}(\bar{\mathcal{T}}_{y,h})_b\right\} +$$

$$+ \overline{\mathcal{E}_b Q}\,\nabla_h H -$$

$$- \nabla_h \overline{\mathcal{E}^{(i)}} +$$

$$+ \overline{(\bar{P} - \bar{E})\hat{\mathbf{v}}_{h,s}}$$

where the different "forces" have been arranged in such a way, that each column is supposed to have a different order of magnitude for purposes regarding the storm surge. Unfortunately too little is known about the order of magnitude to be sure of this arrangement. Also, a change of terms from one column to the other is possible during the approach of a storm to the coast.

It is interesting to formulate the equation of continuity for an incompressible fluid which replaces (41) in this case. Averaging (27") over time leads to

$$\frac{\partial(\bar{\bar{\eta}}+H)}{\partial t} + \nabla_h \cdot \left\{ \overline{(\bar{\eta}+H)\mathbb{V}_h} \right\} = \left(\frac{\overline{\bar{P}-\bar{E}}}{\bar{\rho}_s} \right) + \nabla_h \cdot \left\{ \overline{(\bar{\eta}+H)\hat{\mathbf{V}}_h^{**}} \right\}$$

or

$$\frac{\partial(\bar{\bar{\eta}}+H)}{\partial t} + \nabla_h \cdot \left\{ (\bar{\bar{\eta}}+H)\hat{\mathbb{V}}_h \right\} = \left(\frac{\overline{\bar{P}-\bar{E}}}{\bar{\rho}_s} \right) + \nabla_h \cdot \left\{ \overline{(\bar{\eta}+H)\hat{\mathbf{V}}_h^{**}} - \overline{\bar{\eta}'\mathbb{V}_h''} \right\} \qquad (41')$$

which gives the change of the total water depth at a fixed point in terms of the divergence of the mean volume transport vector $(\bar{\bar{\eta}}+H)\hat{\mathbb{V}}_h$ and some sources or sinks for $\bar{\bar{\eta}}+H$ on the right hand side including besides the time mean of the term depending on the integrated deviations $\hat{\mathbf{V}}_h^{**}$ from the weighted vertical mean and also on the correlation between the heights of the superimposed smaller surface waves and the corresponding fluctuations of the mean velocity vector $\mathbb{V}_h$.

Introducing $\bar{Q} = \overline{\bar{\rho}\,(\bar{\eta}+H)} = \bar{\bar{\rho}}\,(\bar{\bar{\eta}}+H) + \overline{\bar{\rho}'\bar{\eta}'}$ in the general equation of continuity (41) this equation can be written in the form

$$\frac{\partial(\bar{\bar{\eta}}+H)}{\partial t} + \nabla_h \cdot \left\{ (\bar{\bar{\eta}}+H)\hat{\mathbb{V}}_h \right\} = \frac{\bar{\bar{P}}-\bar{\bar{E}}}{\bar{\bar{\rho}}} - \frac{\bar{\bar{\eta}}+H}{\bar{\bar{\rho}}}\left\{ \frac{\partial\bar{\bar{\rho}}}{\partial t} + \hat{\mathbb{V}}_h \cdot \nabla_h \bar{\bar{\rho}} \right\} - \frac{1}{\bar{\bar{\rho}}}\left\{ \frac{\partial(\overline{\bar{\rho}'\bar{\eta}'})}{\partial t} + \nabla_h \cdot \left\{ (\overline{\bar{\rho}'\bar{\eta}'})\hat{\mathbb{V}}_h \right\} \right\}$$

from which together with (38') (for incompressible fluids) the condition for incompressibility for the mean motion follows in the form

$$\frac{\partial\bar{\bar{\rho}}}{\partial t} + \hat{\mathbb{V}}_h \cdot \nabla_h \bar{\bar{\rho}} = \frac{1}{\bar{\bar{\eta}}+H}\left[(\bar{\bar{P}}-\bar{\bar{E}}) - \bar{\bar{\rho}}\left(\frac{\overline{\bar{P}-\bar{E}}}{\bar{\rho}_s}\right) - \bar{\bar{\rho}}\nabla_h \cdot \left\{ \overline{(\bar{\eta}+H)\hat{\mathbf{V}}_h^{**}} - \overline{\bar{\eta}'\mathbb{V}_h''} \right\} - \left\{ \frac{\partial(\overline{\bar{\rho}'\bar{\eta}'})}{\partial t} + \nabla_h \cdot \left\{ (\overline{\bar{\rho}'\bar{\eta}'})\hat{\mathbb{V}}_h \right\} \right\} \right]$$

The density variations $\bar{\rho}'$ during the interval of averaging in an incompressible fluid correspond to the inhomogeneity of the fluid. These density variations certainly are small in most cases and therefore the last term on the right hand side can be neglected.

Disregarding further vertical density variations and the precipitation-minus-"evaporation" effect the condition of incompressibility can be used in the form

$$\frac{\partial\overline{\overline{\overline{\rho}}}}{\partial t} + \hat{\mathbf{V}}_h \cdot \nabla_h \overline{\overline{\overline{\rho}}} = \frac{\overline{\overline{\overline{\rho}}}}{\overline{\overline{\eta}}+H} \nabla_h \cdot \left| \overline{\overline{\eta}'\mathbf{V}_h''} \right| \approx 0$$

For general considerations, however, it is more convenient to use equation (41) instead of (41') and introduce the property of incompressibility at a later stage of the calculations.

5.3 The homogeneous ocean and frequently used storm surge equations

In most of the mathematical treatments of the storm surge the assumptions of homogeneity and incompressibility are made. These assumptions mean in our notations

$$\rho = \overline{\rho} = \overline{\overline{\rho}} = \overline{\overline{\overline{\rho}}} = \text{const.}$$
$$\nabla_h \overline{\overline{\overline{\rho}}} = 0. \tag{56}$$

Therefore we get a system of equations for the homogeneous ocean by dividing (41) and (55) by ρ. Introducing further the definition (35) for $\eta^{(a)}$ we have

$$\frac{\partial(\overline{\overline{\eta}}+H)}{\partial t} + \nabla_h \cdot \left\{ (\overline{\overline{\eta}}+H)\hat{\mathbf{V}}_h \right\} = \frac{\overline{\overline{P}}-\overline{\overline{E}}}{\rho} \tag{57}$$

$$\frac{\partial}{\partial t}\left| (\overline{\overline{\eta}}+H)\hat{\mathbf{V}}_h \right| + \nabla_h \cdot \left| (\overline{\overline{\eta}}+H)\hat{\mathbf{V}}_h\hat{\mathbf{V}}_h \right| + f\mathbf{k}\times(\overline{\overline{\eta}}+H)\hat{\mathbf{V}}_h - \nabla_h \cdot \frac{\overline{\mathcal{F}}_h}{\rho} =$$

$$= (\overline{\overline{\eta}}+H)\left\{ \frac{\hat{d}_h\hat{\mathbf{V}}_h}{dt} + f\mathbf{k}\times\hat{\mathbf{V}}_h + \frac{\overline{\overline{P}}-\overline{\overline{E}}}{\rho(\overline{\overline{\eta}}+H)}\hat{\mathbf{V}}_h \right\} - \nabla_h \cdot \frac{\overline{\mathcal{F}}_h}{\rho} =$$

$$\begin{aligned}
= & -g(\overline{\overline{\eta}}+H)\nabla_h \overline{\overline{\eta}} - g\,\nabla_h\left(\frac{\overline{\overline{\eta}'^2}}{2}\right) + g(\overline{\overline{\eta}}+H)\nabla_h \eta^{(T)} \\
& + g(\overline{\overline{\eta}}+H)\nabla_h \overline{\overline{\eta}}^{(a)} + g\,\overline{\overline{\eta}'\nabla_h\overline{\overline{\eta}}'^{(a)}} \\
& + \frac{\overline{\mathcal{T}}^{(a)}_{z,h}}{\rho} - \overline{\nabla_h\overline{\overline{\eta}}' \cdot \left(\mathbf{i}\frac{\mathcal{T}'^{(a)}_{x,h}}{\rho} + \mathbf{j}\frac{\mathcal{T}'^{(a)}_{y,h}}{\rho}\right)} - \nabla_h\overline{\overline{\eta}} \cdot \left(\mathbf{i}\frac{\overline{\mathcal{T}}^{(a)}_{x,h}}{\rho} + \mathbf{j}\frac{\overline{\mathcal{T}}^{(a)}_{y,h}}{\rho}\right) \\
& - \frac{(\overline{\mathcal{T}}_{z,h})_b}{\rho} - \nabla_h H \cdot \left(\mathbf{i}\frac{(\overline{\mathcal{T}}_{x,h})_b}{\rho} + \mathbf{j}\frac{(\overline{\mathcal{T}}_{y,h})_b}{\rho}\right) \\
& + \overline{\mathcal{E}_b'\,\overline{\overline{\eta}}'}\,\nabla_h H + (\overline{\overline{\eta}}+H)\,\overline{\mathcal{E}}_b\,\nabla_h H \\
& - \nabla_h \frac{\overline{\mathcal{E}^{(i)}}}{\rho} \\
& + \overline{\frac{\overline{P}-\overline{E}}{\rho}\hat{\mathbf{V}}_{h,s}}
\end{aligned} \tag{58}$$

which system can be considered as a system for the volume transport vector $(\bar{\bar{\eta}}+H)\hat{\mathbb{V}}_h$ and the water elevation $\bar{\bar{\eta}}$, when the remaining variables are known.

Sometimes it is convenient to introduce into (57), (58) the speed of long waves C_L and a fictitious speed C_s, defined by

$$C_L^2 = g(\bar{\bar{\eta}}+H) \ ; \qquad C_s^2 = g(\bar{\eta}^{(a)}+H)$$

which leads to the system

$$\hat{d}_h \frac{(2C_L)}{dt} + C_L \nabla_h \cdot \hat{\mathbb{V}}_h = \frac{g}{\rho C_L}(\bar{\bar{P}}-\bar{\bar{E}}) \tag{60}$$

$$\begin{aligned}
&\frac{\hat{d}_h \hat{\mathbb{V}}_h}{dt} + f\mathbf{k}\times\hat{\mathbb{V}}_h - \frac{g}{\rho C_L^2}(\bar{\bar{P}}-\bar{\bar{E}})(\hat{\bar{\mathbf{v}}}_{h,s} - \hat{\mathbb{V}}_h) - \frac{g}{\rho C_L^2}\nabla_h \cdot \bar{\mathcal{F}}_h = \\
&= - C_L \nabla_h (2C_L) \qquad - \frac{g^2}{C_L^2}\nabla_h \left(\frac{\overline{\eta'^2}}{2}\right) \qquad + g\nabla_h \eta^{(T)} \\
&+ C_s \nabla_h (2C_s) \qquad\qquad + \frac{g^2}{C_L^2}\overline{\eta' \nabla_h \eta'^{(a)}} \\
&+ \frac{g}{\rho C_L^2}\bar{\mathcal{T}}^{(a)}_{z,h} - \frac{g}{\rho C_L^2}\overline{\nabla_h \eta' \cdot (\mathbf{i}\,\mathcal{T}'^{(a)}_{x,h} + \mathbf{j}\,\mathcal{T}'^{(a)}_{y,h})} \qquad - \frac{1}{\rho C_L}\nabla_h(2C_L)\cdot(\mathbf{i}\,\bar{\mathcal{T}}^{(a)}_{x,h} + \mathbf{j}\,\bar{\mathcal{T}}^{(a)}_{y,h}) \\
&- \frac{g}{\rho C_L^2}(\bar{\mathcal{T}}_{z,h})_b \qquad + \frac{g}{\rho C_L^2}\nabla_h H \cdot \left\{(\mathbf{i}\,\bar{\mathcal{T}}^{(a)}_{x,h} + \mathbf{j}\,\bar{\mathcal{T}}^{(a)}_{y,h}) - \left(\mathbf{i}(\bar{\mathcal{T}}_{x,h})_b + \mathbf{j}(\bar{\mathcal{T}}_{y,h})_b\right)\right\} \\
&+ \frac{g}{C_L^2}\overline{\mathcal{E}_b' \eta'}\,\nabla_h H \qquad + \bar{\mathcal{E}}_b \nabla_h H \\
&= \frac{g}{\rho C_L^2}\nabla_h \bar{\mathcal{E}}^{(i)} + \frac{g}{\rho C_L^2}\overline{(\bar{P}'-\bar{E}')\hat{\mathbf{v}}'_{h,s}}
\end{aligned} \tag{61}$$

a system for the field variables $\hat{\mathbb{V}}_h$ and C_L.

5.31 Equations in connection with the first approximation of the shallow-water wave theory

If for the long waves we assume hydrostatic equilibrium holds in the mean and if in the friction tensor $\bar{\mathcal{F}}_h$ we neglect second order non-linear terms which were the result of vertical averaging, then the terms

$$\bar{\mathcal{E}}_b \nabla_h H \ , \quad -\nabla_h \bar{\mathcal{E}}^{(i)} \qquad \text{and} \qquad (\bar{\bar{\eta}}+H)\overline{(-\bar{\rho}\,\hat{\mathbf{v}}_h^{**}\hat{\mathbf{v}}_h^{**})}$$

drop out and the resulting equations may be called equations of first approximation in analogy to the corresponding equations in the non-linear shallow-water wave theory. The basic equations of this theory are obtained from (57, 58 or 60, 61) by neglecting all external influences and internal friction together with the assumption that no small surface waves are present; i.e., (Wehausen and Laitone [38] and Stoker [29])

$$\frac{\partial(\bar{\bar{\eta}}+H)}{\partial t} + \nabla_h \cdot \left| (\bar{\bar{\eta}}+H)\hat{\mathbb{V}}_h \right| = 0$$

$$\frac{\partial}{\partial t}\left\{(\bar{\bar{\eta}}+H)\hat{\mathbb{V}}_h\right\} + \nabla_h \cdot \left\{(\bar{\bar{\eta}}+H)\hat{\mathbb{V}}_h\hat{\mathbb{V}}_h\right\} = -g(\bar{\bar{\eta}}+H)\nabla_h\bar{\bar{\eta}} \tag{62}$$

or

$$\frac{\hat{d}_h(2C_L)}{dt} + C_L\nabla_h \cdot \hat{\mathbb{V}}_h = 0$$

$$\frac{\hat{d}_h\hat{\mathbb{V}}_h}{dt} = -C_L\nabla_h(2C_L) + g\nabla_h H \tag{63}$$

The extension of these equations to equations including storm surge effects and the Coriolis acceleration but excluding the effects of superimposed surface waves and the terms in the third column of equations (58) and (61) is given by Freeman and Baer [13].

$$\frac{\partial(\bar{\bar{\eta}}+H)}{\partial t} + \nabla_h \cdot \left| (\bar{\bar{\eta}}+H)\hat{\mathbb{V}}_h \right| = 0$$

$$\frac{\partial}{\partial t}\left|(\bar{\bar{\eta}}+H)\hat{\mathbb{V}}_h\right| + \nabla_h \cdot \left|(\bar{\bar{\eta}}+H)\hat{\mathbb{V}}_h\hat{\mathbb{V}}_h\right| + f\mathbf{k}\times(\bar{\bar{\eta}}+H)\hat{\mathbb{V}}_h = \tag{64}$$

$$= -g(\bar{\bar{\eta}}+H)\nabla_h(\bar{\bar{\eta}} - \bar{\eta}^{(a)}) + \frac{1}{\rho}\left(\bar{\bar{\tau}}^{(a)}_{z,h} - (\bar{\bar{\tau}}_{z,h})_b\right)$$

or

$$\frac{\hat{d}_h(2C_L)}{dt} + C_L\nabla_h \cdot \hat{\mathbb{V}}_h = 0$$

$$\frac{\hat{d}_h\hat{\mathbb{V}}_h}{dt} + f\mathbf{k}\times\hat{\mathbb{V}}_h = -C_L\nabla_h(2C_L) + C_s\nabla_h(2C_s) + \frac{g}{\rho C_L^2}\left(\bar{\tau}^{(a)}_{z,h} - (\bar{\tau}_{z,h})_b\right) \tag{65}$$

These can be solved by the method of wave-derivatives (Freeman and Baer [11, 13]). Lateral friction is still neglected in these systems. However, the systems are closed for the unknown functions if the stress at the bottom is known.

5.32 Separation of a basic current

It is desirable, in storm surge research to separate from the total solution a basic state which would be present in the absence of the storm. This basic current includes the astronomical tide, semipermanent currents, and any transient currents not due to the storm being investigated. The Gulf Stream and the counter currents near the shore are examples of semipermanent and transient currents.

The complete equations (57), (58) for the homogeneous ocean in abridged form may be stated as

$$\frac{\partial(\bar{\bar{\eta}}+H)}{\partial t} + \nabla_h \cdot \left[(\bar{\bar{\eta}}+H)\hat{\mathbb{V}}_h\right] = \frac{\bar{\bar{P}}-\bar{\bar{E}}}{\rho}$$

$$\frac{\partial\hat{\mathbb{V}}_h}{\partial t} + \hat{\mathbb{V}}_h \cdot \nabla_h\hat{\mathbb{V}}_h + f\mathbf{k}\times\hat{\mathbb{V}}_h + \frac{\bar{\bar{P}}-\bar{\bar{E}}}{\rho}\hat{\mathbb{V}}_h = -g\nabla_h(\bar{\bar{\eta}} - \bar{\eta}^{(a)} - \eta^{(T)})$$

$$+\frac{1}{\rho(\bar{\bar{\eta}}+H)}\left\{\nabla_h \cdot \bar{\mathcal{F}}_h + \bar{\tau}^{(a)}_{z,h} - (\bar{\tau}_{z,h})_b - \nabla_h H \cdot (\mathbf{i}(\bar{\tau}_{x,h})_b + \mathbf{j}(\bar{\tau}_{y,h})_b) + \rho\mathbf{K}_h\right\}$$

where

$$\mathbf{K}_h = \overline{-g\nabla_h\left(\frac{\bar{\eta}'^2}{2}\right) - \nabla_h\bar{\eta}' \cdot \left(\mathbf{i}\frac{\tau'^{(a)}_{x,h}}{\rho} + \mathbf{j}\frac{\tau'^{(a)}_{y,h}}{\rho}\right) + (\overline{\mathcal{E}_b'\bar{\eta}'})\nabla_h H}$$

$$\overline{+g\bar{\eta}'\nabla_h\bar{\eta}'^{(a)} - \nabla_h\bar{\bar{\eta}} \cdot \left(\mathbf{i}\frac{\bar{\tau}^{(a)}_{x,h}}{\rho} + \mathbf{j}\frac{\bar{\tau}^{(a)}_{y,h}}{\rho}\right) + \bar{\mathcal{E}}_b(\bar{\bar{\eta}}+H)\nabla_h H - \nabla_h\frac{\overline{\mathcal{E}^{(i)}}}{\rho} + \frac{\bar{\bar{P}}-\bar{\bar{E}}}{\rho}\hat{\mathbb{V}}_{h,s}}$$

is assumed to vanish for the basic current. The basic current is defined by the following (hydrostatically approximated) system

$$\frac{\partial(\bar{\bar{\eta}}^{(o)}+H)}{\partial t} + \nabla_h \cdot \left[(\bar{\bar{\eta}}^{(o)}+H)\hat{\mathbb{V}}_h^{(o)}\right] = 0 \tag{66}$$

$$\frac{\partial\hat{\mathbb{V}}_h^{(o)}}{\partial t} + \hat{\mathbb{V}}_h^{(o)} \cdot \nabla_h\hat{\mathbb{V}}_h^{(o)} + f\mathbf{k}\times\hat{\mathbb{V}}_h^{(o)} = -g\nabla_h(\bar{\bar{\eta}}^{(o)} - \eta^{(T)})$$

$$+\frac{1}{\rho(\bar{\bar{\eta}}^{(o)}+H)}\left\{\nabla_h \cdot \bar{\mathcal{F}}_h^{(o)} - (\bar{\tau}^{(o)}_{z,h})_b - \nabla_h H \cdot (\mathbf{i}(\bar{\tau}^{(o)}_{x,h})_b + \mathbf{j}(\bar{\tau}^{(o)}_{y,h})_b)\right\} \tag{67}$$

which does not contain any storm effects. If one separates the storm induced variables from the basic ones by

$$\hat{\mathbb{V}}_h = \hat{\mathbb{V}}_h^{(o)} + \hat{\mathbb{V}}_h^{(*)} = \qquad \bar{\bar{\eta}} = \bar{\bar{\eta}}^{(o)} + \bar{\bar{\eta}}^{(*)} \tag{68}$$

the pure storm surge equations for the homogeneous ocean may be stated in the form

$$\frac{\partial\bar{\bar{\eta}}^{(*)}}{\partial t} + \nabla_h \cdot \left[(\bar{\bar{\eta}}^{(o)}+H)\hat{\mathbb{V}}_h^{(*)} + \bar{\bar{\eta}}^{(*)}\hat{\mathbb{V}}_h^{(o)} + \bar{\bar{\eta}}^{(*)}\hat{\mathbb{V}}_h^{(*)}\right] = \frac{\bar{\bar{P}}-\bar{\bar{E}}}{\rho} \tag{69}$$

$$\frac{\partial\hat{\mathbf{V}}_h^{(*)}}{\partial t}+(\hat{\mathbf{V}}_h^{(o)}+\hat{\mathbf{V}}_h^{(*)})\cdot\nabla_h\hat{\mathbf{V}}_h^{(*)}+\hat{\mathbf{V}}_h^{(*)}\cdot\nabla_h\hat{\mathbf{V}}_h^{(o)}+f\mathbf{k}\times\hat{\mathbf{V}}_h^{(*)}+\frac{\overline{\overline{P}}-\overline{\overline{E}}}{\rho}(\hat{\mathbf{V}}_h^{(o)}+\hat{\mathbf{V}}_h^{(*)})$$

$$=-g\nabla_h(\overline{\overline{\eta}}^{(*)}-\overline{\eta}^{(a)})$$

$$+\frac{1}{\rho(\overline{\overline{\eta}}^{(o)}+\overline{\overline{\eta}}^{(*)}+H)}\left\{\nabla_h\cdot\overline{\mathfrak{F}}_h^{(*)}+\overline{\mathfrak{T}}_{z,h}^{(a)}-(\overline{\mathfrak{T}}_{z,h}^{(*)})_b-\nabla_h H\cdot(\mathbf{i}(\overline{\mathfrak{T}}_{x,h}^{(*)})_b+\mathbf{j}(\overline{\mathfrak{T}}_{y,h}^{(*)})_b)+\rho\mathbf{K}_h\right\} \tag{70}$$

$$-\frac{\overline{\overline{\eta}}^{(*)}}{\rho(\overline{\overline{\eta}}^{(o)}+H)(\overline{\overline{\eta}}^{(o)}+\overline{\overline{\eta}}^{(*)}+H)}\left\{\nabla_h\cdot\overline{\mathfrak{F}}_h^{(o)}-(\overline{\mathfrak{T}}_{z,h}^{(o)})_b-\nabla_h H\cdot(\mathbf{i}(\overline{\mathfrak{T}}_{x,h}^{(o)})_b+\mathbf{j}(\overline{\mathfrak{T}}_{y,h}^{(o)})_b)\right\}.$$

The storm surge variables, indicated by the star, may not be small compared with the variables of the basic current. Therefore, the interactions between basic current and storm surge current are quite complicated and it will not always be possible to obtain satisfactory results by perturbation methods.

In principle the solution of the basic system (66), (67) has to be known with satisfactory accuracy for each individual domain of the ocean to calculate the required storm surge under given atmospheric conditions.

Disregarding internal lateral friction and neglecting the terms containing $\overline{\overline{P}}-\overline{\overline{E}}$, $\mathbf{K}_h$, and $\nabla_h H\cdot(\overline{\overline{\rho\mathbf{v}_h''\mathbf{v}_h''}})_b$, the simplified system is then given by

$$\frac{\partial\overline{\overline{\eta}}^{(*)}}{\partial t}+\nabla_h\cdot\left\{(\overline{\overline{\eta}}^{(o)}+H)\hat{\mathbf{V}}_h^{(*)}+\overline{\overline{\eta}}^{(*)}(\hat{\mathbf{V}}_h^{(o)}+\hat{\mathbf{V}}_h^{(*)})\right\}=0$$

$$\frac{\partial\hat{\mathbf{V}}_h^{(*)}}{\partial t}+(\hat{\mathbf{V}}_h^{(o)}+\hat{\mathbf{V}}_h^{(*)})\cdot\nabla_h\hat{\mathbf{V}}_h^{(*)}+\hat{\mathbf{V}}_h^{(*)}\cdot\nabla_h\hat{\mathbf{V}}_h^{(o)}+f\mathbf{k}\times\hat{\mathbf{V}}_h^{(*)}=-g\nabla_h(\overline{\overline{\eta}}^{(*)}-\overline{\eta}^{(a)}) \tag{71}$$

$$+\frac{1}{\rho(\overline{\overline{\eta}}+H)}\left\{\overline{\mathfrak{T}}_{z,h}^{(a)}-(\overline{\mathfrak{T}}_{z,h}^{(*)})_b+\frac{\overline{\overline{\eta}}^{(*)}}{\overline{\overline{\eta}}^{(o)}+H}(\overline{\mathfrak{T}}_{z,h}^{(o)})_b\right\}.$$

Another system for the pure storm surge current is obtained if the complete equations (57), (58) are written in the form

$$\frac{\partial(\overline{\overline{\eta}}+H)}{\partial t}+\nabla_h\cdot\mathbf{S}_h=\frac{\overline{\overline{P}}-\overline{\overline{E}}}{\rho} \tag{72}$$

$$\frac{\partial\mathbf{S}_h}{\partial t}+\nabla_h\cdot\left\{\frac{\mathbf{S}_h\mathbf{S}_h}{\overline{\overline{\eta}}+H}\right\}+f\mathbf{k}\times\mathbf{S}_h=-g(\overline{\overline{\eta}}+H)\nabla_h(\overline{\overline{\eta}}-\overline{\eta}^{(a)}-\eta^{(T)})$$

$$+\nabla_h\cdot\frac{\overline{\mathfrak{F}}_h}{\rho}+\frac{1}{\rho}\left\{\overline{\mathfrak{T}}_{z,h}^{(a)}-(\overline{\mathfrak{T}}_{z,h})_b\right\}-\nabla_h H\cdot\left\{\mathbf{i}\frac{(\overline{\mathfrak{T}}_{x,h})_b}{\rho}+\mathbf{j}\frac{(\overline{\mathfrak{T}}_{y,h})_b}{\rho}\right\}+\mathbf{K}_h \tag{73}$$

where the volume transport vector

$$\mathbf{S}_h = (\bar{\bar{\eta}}+H)\hat{\mathbb{V}}_h \tag{74}$$

has been introduced.

Writing the basic current equations also in terms of the volume transport vector

$$\frac{\partial(\bar{\bar{\eta}}^{(o)}+H)}{\partial t} + \nabla_h \cdot \mathbf{S}_h^{(o)} = 0 \tag{75}$$

$$\frac{\partial \mathbf{S}_h^{(o)}}{\partial t} + \nabla_h \cdot \left\{ \frac{\mathbf{S}_h^{(o)}\mathbf{S}_h^{(o)}}{\bar{\bar{\eta}}^{(o)}+H} \right\} + f\mathbf{k}\times\mathbf{S}_h^{(o)} = -g(\bar{\bar{\eta}}^{(o)}+H)\nabla_h(\bar{\bar{\eta}}^{(o)}-\eta^{(T)})$$
$$+\nabla_h \cdot \frac{\bar{\bar{\mathfrak{F}}}_h^{(o)}}{\rho} - \frac{1}{\rho}(\bar{\mathcal{T}}_{z,h}^{(o)})_b - \nabla_h H \cdot \left\{ \mathbf{i}\frac{(\bar{\mathcal{T}}_{x,h}^{(o)})_b}{\rho} + \mathbf{j}\frac{(\bar{\mathcal{T}}_{y,h}^{(o)})}{\rho} \right\} \tag{76}$$

and subtracting this system from (72) and (73) one gets for the storm surge variables $\bar{\bar{\eta}}^{(*)}$ and $\mathbf{S}_h^{(*)}$ the system

$$\frac{\partial\bar{\bar{\eta}}^{(*)}}{\partial t} + \nabla_h \cdot \mathbf{S}_h^{(*)} = \frac{\bar{\bar{P}}-\bar{\bar{E}}}{\rho} \tag{77}$$

$$\frac{\partial \mathbf{S}_h^{(*)}}{\partial t} + \nabla_h \cdot \left\{ \frac{\mathbf{S}_h^{(o)}\mathbf{S}_h^{(*)}+\mathbf{S}_h^{(*)}\mathbf{S}_h^{(o)}+\mathbf{S}_h^{(*)}\mathbf{S}_h^{(*)}}{\bar{\bar{\eta}}+H} - \frac{\bar{\bar{\eta}}^{(*)}\mathbf{S}_h^{(o)}\mathbf{S}_h^{(o)}}{(\bar{\bar{\eta}}+H)(\bar{\bar{\eta}}^{(o)}+H)} \right\} + f\mathbf{k}\times\mathbf{S}_h^{(*)} =$$
$$= -g(\bar{\bar{\eta}}+H)\nabla_h(\bar{\bar{\eta}}^{(*)}-\bar{\eta}^{(a)}) - g\bar{\bar{\eta}}^{(*)}\nabla_h(\bar{\bar{\eta}}^{(o)}-\eta^{(T)}) + \nabla_h \cdot \frac{\bar{\bar{\mathfrak{F}}}_h^{(*)}}{\rho} + \frac{1}{\rho}\left\{ \bar{\mathcal{T}}_{z,h}^{(a)} - (\bar{\mathcal{T}}_{z,h}^{(*)})_b \right\}$$
$$- \nabla_h H \cdot \left\{ \mathbf{i}\frac{(\bar{\mathcal{T}}_{x,h}^{(*)})_b}{\rho} + \mathbf{j}\frac{(\bar{\mathcal{T}}_{y,h}^{(*)})_b}{\rho} \right\} + \mathbf{K}_h, \tag{78}$$

where the term $g\bar{\bar{\eta}}^{(*)}\nabla_h\eta^{(T)}$ is negligible in most cases.

The simplified system corresponding to (71) is given by

$$\left.\begin{aligned}
&\frac{\partial\bar{\bar{\eta}}^{(*)}}{\partial t} + \nabla_h \cdot \mathbf{S}_h^{(*)} = 0 \\
&\frac{\partial \mathbf{S}_h^{(*)}}{\partial t} + \nabla_h \cdot \left\{ \frac{\mathbf{S}_h\mathbf{S}_h}{\bar{\bar{\eta}}+H} - \frac{\mathbf{S}_h^{(o)}\mathbf{S}_h^{(o)}}{\bar{\bar{\eta}}^{(o)}+H} \right\} + f\mathbf{k}\times\mathbf{S}_h^{(*)} = \\
&= -g(\bar{\bar{\eta}}+H)\nabla_h(\bar{\bar{\eta}}^{(*)}-\bar{\eta}^{(a)}) - g\,\bar{\bar{\eta}}^{(*)}\nabla_h(\bar{\bar{\eta}}^{(o)}-\eta^{(T)}) + \frac{1}{\rho}\left\{ \bar{\mathcal{T}}_{z,h}^{(a)} - (\bar{\mathcal{T}}_{z,h}^{(*)})_b \right\}
\end{aligned}\right\} \tag{79}$$

where the divergence term on the left hand side of the second equation has been written in the original form.

These surge equations contain in a quite general sense all interactions between the pure surge current, caused by a hurricane acting on the surface of the ocean, and a basic current which includes also the astronomical tidal current. Whether the system (69), (70) or the system (77), (78) is more useful for the study of non-linear interactions cannot be said at this time. However the simple form of the equation of continuity (77) is one advantage of the latter system.

In a linear treatment no interactions can occur. Thus the best approximation at the coast is a simple addition of surge height and tide height. (See equations (83),(84).)

5.33 Equations in connection with linear theories

The starting point of all linear surge theories is the equations for the first approximation of the non-linear shallow water wave theory, i.e., the assumption of hydrostatic equilibrium and the assumption that the horizontal velocity components are not dependent on depth (see 5.31).

In the first step of linearization one usually neglects the non-linear terms which contain the dyadic product of the mean velocity $\hat{\mathbf{V}}_h$ and the terms $\nabla_h\bar{\bar{\eta}} \cdot (\overline{\overline{-\mathbf{V}_h''\mathbf{V}_h''}})^{(a)}, \nabla H \cdot (\overline{\overline{-\rho\mathbf{V}_h''\mathbf{V}_h''}})_b$ on the right hand side of equation (58). This gives the system,

$$\frac{\partial(\bar{\bar{\eta}}+H)}{\partial t} + \nabla_h \cdot \mathbf{S}_h = \frac{\bar{\bar{P}}-\bar{\bar{E}}}{\rho} \tag{72}$$

$$\frac{\partial \mathbf{S}_h}{\partial t} + f\mathbf{k}\times\mathbf{S}_h - \nabla_h \cdot \frac{\bar{\mathfrak{F}}_h^{(1)}}{\rho} = - g(\bar{\bar{\eta}}+H)\nabla_h(\bar{\bar{\eta}}-\bar{\eta}^{(a)}-\eta^{(T)}) + \frac{1}{\rho}\left\{\bar{\mathfrak{T}}_{z,h}^{(a)}-(\mathfrak{T}_{z,h})_b\right\}$$
$$- g\nabla_h\left(\frac{\overline{\bar{\eta}'^2}}{2}\right) + g\,\overline{\bar{\eta}'\,\nabla_h\bar{\eta}'^{(a)}} - \overline{\nabla_h\bar{\eta}' \cdot \left(\mathbf{i}\frac{\mathfrak{T}'^{(a)}_{x,h}}{\rho} + \mathbf{j}\frac{\mathfrak{T}'^{(a)}_{y,h}}{\rho}\right)} + (\overline{\varepsilon_b'\bar{\eta}'})\nabla_h H \tag{80}$$

in terms of the volume transport vector $\mathbf{S}_h$.

Disregarding the effects of superimposed wind-waves and swell, the effect of the excess of precipitation over evaporation, and assuming an approximate expression (48) for the divergence of the friction stress one gets instead of (72), (80) the simplified system

$$\frac{\partial(\bar{\bar{\eta}}+H)}{\partial t} + \nabla_h \cdot \mathbf{S}_h = 0 \tag{81}$$

$$\frac{\partial \mathbf{S}_h}{\partial t} + f\mathbf{k}\times\mathbf{S}_h - K_h\nabla^2\mathbf{S}_h = -g(\bar{\bar{\eta}}+H)\nabla_h(\bar{\bar{\eta}}-\bar{\eta}^{(a)}-\eta^{(T)}) + \frac{1}{\rho}\left\{\bar{\mathfrak{T}}_{z,h}^{(a)} - (\mathfrak{T}_{z,h})_b\right\} \tag{82}$$

or, with the further approximation $\bar{\eta} \ll H$ and the assumption $H(x,y,t) = \bar{H}(x,y)$

$$\frac{\partial \bar{\bar{\eta}}}{\partial t} + \nabla_h \cdot \mathbf{S}_h = 0 \tag{83}$$

$$\frac{\partial \mathbf{S}_h}{\partial t} + f\mathbf{k}\times\mathbf{S}_h - K_h \nabla_h^2 \mathbf{S}_h = -gH\nabla_h(\bar{\bar{\eta}}-\bar{\eta}^{(a)}-\eta^{(T)}) + \frac{1}{\rho}\left\{\bar{\mathcal{T}}_{z,h}^{(a)} - (\bar{\mathcal{T}}_{z,h})_b\right\}_o \tag{84}$$

This system is in common use for almost all treatments of oceanographical problems using the vertically integrated equations of motions and is used in almost all theoretical considerations in connection with the storm surge problem.

5.34 Wave equations

The complete system for the homogeneous ocean (57), (58) can be written in the form

$$-\nabla_h \cdot \mathbf{S}_h = \frac{\partial(\bar{\bar{\eta}}+H)}{\partial t} - \frac{\bar{\bar{P}}-\bar{\bar{E}}}{\rho} \tag{85}$$

$$\frac{\partial \mathbf{S}_h}{\partial t} + \alpha\mathbf{S}_h - K_h\nabla_h^2\mathbf{S}_h + f\mathbf{k}\times\mathbf{S}_h = -\mathbb{F}_h \tag{86}$$

where some "forces" have been split up and where now

$$\begin{aligned}
-\mathbb{F}_h = {} & -\nabla_h \cdot \left\{\frac{\mathbf{S}_h\mathbf{S}_h}{\bar{\bar{\eta}}+H}\right\} - g(\bar{\bar{\eta}}+H)\nabla_h(\bar{\bar{\eta}}-\bar{\eta}^{(a)}-\eta^{(T)}) + \nabla_h \cdot \frac{\bar{\mathcal{F}}_h}{\rho} - K_h\nabla_h^2\mathbf{S}_h \\
& + \frac{1}{\rho}\left\{\bar{\mathcal{T}}_{z,h}^{(a)} - \left((\bar{\mathcal{T}}_{z,h})_b - \rho\alpha\mathbf{S}_h\right)\right\} - g\nabla_h\left(\frac{\overline{\eta'^2}}{2}\right) - \overline{\nabla_h\eta'\left(\mathbf{i}\frac{\mathcal{T}'^{(a)}_{x,h}}{\rho} + \mathbf{j}\frac{\mathcal{T}'^{(a)}_{y,h}}{\rho}\right)} \\
& + \overline{\mathcal{E}_b'\eta'}\,\nabla_h H + \overline{g\eta'\nabla_h\eta'^{(a)}} - \nabla_h\bar{\bar{\eta}} \cdot \left(\mathbf{i}\frac{\bar{\mathcal{T}}^{(a)}_{x,h}}{\rho} + \mathbf{j}\frac{\bar{\mathcal{T}}^{(a)}_{y,h}}{\rho}\right) \\
& - \nabla_h H \cdot \left(\mathbf{i}\frac{(\bar{\mathcal{T}}_{x,h})_b}{\rho} + \mathbf{j}\frac{(\bar{\mathcal{T}}_{y,h})_b}{\rho}\right) \\
& + (\bar{\bar{\eta}}+H)\,\bar{\mathcal{E}}_b\nabla_h H - \nabla_h\frac{\overline{\mathcal{E}^{(i)}}}{\rho} + \overline{\frac{\bar{P}-\bar{E}}{\rho}\hat{\mathbf{v}}_{h,s}}
\end{aligned} \tag{87}$$

contains the non-linear terms, the deviation of the lateral friction term from the simple expression (48), a modified bottom stress which is assumed to be proportional to the surface stress, and all correction terms connected with superimposed wind-waves and those discussed in 5.1.

It is possible to derive a wave equation for $\bar{\bar{\eta}}+H$ from (85), (86) provided that the coefficients on the left hand side of (86) are constant. Taking the horizontal divergence and the vertical component of the vorticity of the volume transport vector one obtains

$$\left(\frac{\partial}{\partial t} + \alpha - K_h\nabla_h^2\right)\nabla_h \cdot \mathbf{S}_h - f\mathbf{k}\cdot\nabla_h \times \mathbf{S}_h = -\nabla_h \cdot \mathbb{F}_h \qquad (88)$$

$$\left(\frac{\partial}{\partial t} + \alpha - K_h\nabla_h^2\right)\mathbf{k}\cdot\nabla_h \times \mathbf{S}_h + f\nabla_h \cdot \mathbf{S}_h = -\mathbf{k}\cdot\nabla_h \times \mathbb{F}_h \qquad (89)$$

Applying the operator $\left(\frac{\partial}{\partial t} + \alpha - K_h\nabla_h^2\right)$ on (88) and introducing (89) together with (85) leads to a wave equation for $\bar{\bar{\eta}}+H$, namely

$$\left\{\left(\frac{\partial}{\partial t} + \alpha - K_h\nabla_h^2\right)^2 + f^2\right\}\frac{\partial(\bar{\bar{\eta}}+H)}{\partial t} = \left(\frac{\partial}{\partial t} + \alpha - K_h\nabla_h^2\right)\nabla_h\cdot\mathbb{F}_h + \mathbf{k}\cdot\nabla_h \times \mathbb{F}_h$$
$$+ \left\{\left(\frac{\partial}{\partial t} + \alpha - K_h\nabla_h^2\right)^2 + f^2\right\}\frac{\bar{\bar{P}}-\bar{\bar{E}}}{\rho}. \qquad (90)$$

The wave equations of all simplified models can be derived from this equation by specifying the vector $\mathbb{F}_h$. The simplest possible specification is the linearization and the neglect of all complicated terms i. e.,

$$-\mathbb{F}_h = -gH\nabla_h(\bar{\bar{\eta}}-\bar{\eta}^{(a)}-\eta^{(T)}) + n\bar{\mathcal{T}}^{(a)}_{z,h} \qquad (91)$$

where the assumption

$$(\bar{\mathcal{T}}_{z,h})_b = \rho\alpha\mathbf{S}_h + n\bar{\mathcal{T}}^{(a)}_{z,h} \qquad (92)$$

for the bottom stress has been used.

The principal value of the wave equation (90) is its application to the linearized case where the term $\nabla_h\cdot\left\{\frac{\mathbf{S}_h\mathbf{S}_h}{\bar{\bar{\eta}}+H}\right\}$ can be neglected. In the more general non-linear problems the separation of variables is a much more complicated problem and cannot be carried out in general and therefore no simple wave equation can be expected. However, the number of equations can be reduced by introducing two scalar functions instead of the components of the volume transport vector defined by

$$\mathbf{S}_h = \nabla_h\frac{\partial\chi}{\partial t} + \mathbf{k}\times\nabla_h\psi \qquad (93)$$

Writing

$$\frac{\partial\bar{\bar{\eta}}^{(P-E)}}{\partial t} = \frac{\bar{\bar{P}}-\bar{\bar{E}}}{\rho}$$

introduces the local change with time of the water elevation due to the precipitation minus evaporation effect, and the equation of continuity (85) is fulfilled if we set

$$\bar{\bar{\eta}}+H = -\nabla_h^2\chi + \bar{\bar{\eta}}^{(P-E)} \qquad (94)$$

Introduction of (93) and (94) into (57), (58) or (85), (86) leaves one vectorial equation for the unknown scalar functions χ and ψ.

5.4 Wave equation for the non-homogeneous ocean derived from the general mass transport equation

A wave equation similar to (90) can be derived for the mass transport in an inhomogeneous ocean in a manner analogous to the derivation of (90) by separation of the lateral friction approximation (48) and by introduction of the expression

$$(\overline{\mathcal{T}}_{z,h})_b = \overset{(*)}{\alpha}\,\mathbf{M}_h + n\,\overline{\mathcal{T}}^{(a)}_{z,h} \tag{92'}$$

where the horizontal mass transport vector $\overline{Q}\hat{\mathbb{V}}_h$ is denoted by $\mathbf{M}_h$.

Writing (41), (55) in the form

$$-\nabla_h \cdot \mathbf{M}_h = \frac{\partial \overline{Q}}{\partial t} - (\overline{\overline{P}}-\overline{\overline{E}}) \tag{85'}$$

$$\frac{\partial \mathbf{M}_h}{\partial t} + \overset{(*)}{\alpha}\,\mathbf{M}_h - K_h \nabla_h^2 \mathbf{M}_h + f\mathbf{k}\times\mathbf{M}_h = -\overset{(*)}{\mathbb{F}^{\nabla}_h} \tag{86'}$$

where here the "force" $\overset{(*)}{\mathbb{F}^{\nabla}_h}$ is given by the expression

$$\begin{aligned}
-\overset{(*)}{\mathbb{F}^{\nabla}_h} = {} & -\nabla_h \cdot \left\{\frac{\mathbf{M}_h\mathbf{M}_h}{\overline{Q}}\right\} - g\overline{Q}\,\nabla_h(\overline{\overline{\eta}}-\eta^{(T)}) - (\overline{\overline{\eta}}+H)\,\nabla_h \overline{\overline{p}}^{(a)} + \nabla_h \cdot \overline{\mathcal{F}}_h - K_h \nabla_h^2 \mathbf{M}_h \\
& + (1-\eta)\overline{\mathcal{T}}^{(a)}_{z,h} - g\,\overline{\overline{\overline{\rho}}}\,\nabla_h\left(\frac{\overline{\eta'^2}}{2}\right) - \frac{g}{2}\left|(\overline{\overline{\eta}}+H)^2 + (\overline{\eta'^2})\right|\nabla_h\overline{\overline{\overline{\rho}}} - g\nabla_h\left(\overline{\int_{-H}^{\overline{\eta}} dz \int_z^{\eta} \overline{\rho}^* d\zeta}\right) \\
& - \overline{\nabla_h \overline{\eta}' \cdot (\mathbf{i}\,\mathcal{T}'^{(a)}_{x,h} + \mathbf{j}\,\mathcal{T}'^{(a)}_{y,h})} - \overline{\overline{\eta}'\,\nabla_h \overline{p}'^{(a)}} - g(\overline{\overline{\eta}}+H)\,\nabla_h(\overline{\overline{\overline{\rho}}'\overline{\eta}'}) - g\nabla_h(\overline{\overline{\overline{\rho}}'\overline{\eta}'^2}/2) \\
& - \nabla_h\overline{\overline{\eta}} \cdot (\mathbf{i}\,\overline{\mathcal{T}}^{(a)}_{x,h} + \mathbf{j}\,\overline{\mathcal{T}}^{(a)}_{y,h}) - \nabla_h H \cdot \left|\mathbf{i}(\overline{\mathcal{T}}_{x,h})_b + \mathbf{j}(\overline{\mathcal{T}}_{y,h})_b\right| \\
& + \overline{\mathcal{E}_b Q}\,\nabla_h H - \nabla_h \overline{\mathcal{E}^{(i)}} + \overline{(\overline{P}-\overline{E})\hat{\mathbf{v}}_{h,s}}
\end{aligned} \tag{87'}$$

the "wave" equation similar to (90) follows by comparison of (86) and (86') in the form

$$\begin{aligned}
\left\{\left(\frac{\partial}{\partial t} + \overset{(*)}{\alpha} - K_h\nabla_h^2\right)^2 + f^2\right\}\frac{\partial \overline{Q}}{\partial t} = {} & \left(\frac{\partial}{\partial t} + \overset{(*)}{\alpha} - K_h\nabla_h^2\right)\nabla_h \cdot \overset{(*)}{\mathbb{F}^{\nabla}_h} + \mathbf{k}\cdot\nabla_h\times\overset{(*)}{\mathbb{F}^{\nabla}_h} \\
& + \left|\left(\frac{\partial}{\partial t} + \overset{(*)}{\alpha} - K_h\nabla_h^2\right)^2 + f^2\right|(\overline{\overline{P}}-\overline{\overline{E}})
\end{aligned} \tag{90'}$$

where $\overset{(*)}{\alpha}$ and f are assumed to be constant.

This is a wave equation for the mass $\overline{Q}$ of a vertical column of water in which the highest time derivative appears on the left side. But on the right side, there are still time derivatives, derivatives of $\overline{Q}$, $\overline{\overline{\eta}}$, and of all the other terms which cannot be determined separately in this treatment.

The evaluation of the divergence and the vorticity of the "force" vector $\overset{(*)}{\mathbb{F}}_h$ on the right side of equation (90') is a matter of simple differentiations and will not be given explicitly in this paper. However, it can be seen from (90') that these approximations lead to the different equations which are used in connection with the vertically integrated equations. For example, neglecting the derivative $\frac{\partial \overline{Q}}{\partial t}$ of the mass of a vertical column with time in the equation of continuity and introducing a mass transport stream function, allows us in the derivation of (90') to incorporate the Rossby-parameter β, (i.e., one can take into account variation of f with latitude). Then the equations of Welander [39] are obtained by dropping the left hand side of (90'), neglecting several terms in $\overset{(*)}{\mathbb{F}}_h$, and using instead of the bottom stress assumption (92') the expression

$$(\overline{\mathfrak{T}}_{z,h})_b = - D\left\{\nabla_h \bar{\bar{p}}_b - \mathbf{k}\times\nabla_h\bar{\bar{p}}_b\right\} \quad \text{(with } D = \frac{1}{2\pi} \text{ times Ekman depth)} \tag{92"}$$

and putting $\overset{(*)}{\alpha}$ equal to zero in this case.

Whether the resulting equations which exclude "certain" types of waves due to the assumption $\partial\overline{Q}/\partial t = 0$ should be used for storm surge calculations is uncertain.

5.5 Introduction of bathystrophic coordinates

5.51 Definition of bathystrophic coordinates

We define a bathystrophic coordinate system σ-ν as follows: The first family of coordinate lines is given by the lines of equal depth, the σ-lines. The second family of coordinate lines is arbitrary, but we choose here the ν-lines which are orthogonal to the σ-lines.

Given the lines of equal depth -H = const. in the horizontal plane, a system of unit vectors $\boldsymbol{\sigma}\cdot\boldsymbol{\nu}$ which are orthogonal, with the relationships $\boldsymbol{\nu}\cdot\boldsymbol{\sigma} = 0, \boldsymbol{\nu}^2 = \boldsymbol{\sigma}^2 = 1$, can be defined by

$$\boldsymbol{\nu} = \frac{\nabla_h(-H)}{|\nabla_h(-H)|} = - \frac{1}{|\nabla_h H|}\nabla_h H \tag{95}$$

$$\boldsymbol{\sigma} = \boldsymbol{\nu}\times\mathbf{k} = \frac{1}{|\nabla_h H|}(\mathbf{k}\times\nabla_h H) \tag{96}$$

Therefore $\boldsymbol{\sigma}$ is the tangential vector of the lines -H = const., the σ-lines, and directed so that it has decreasing depth on the left hand side. The vector $\boldsymbol{\nu}$ is the normal vector of the σ-lines and the tangential vector of the ν-lines and is pointed toward decreasing depth. (See fig. 2.)

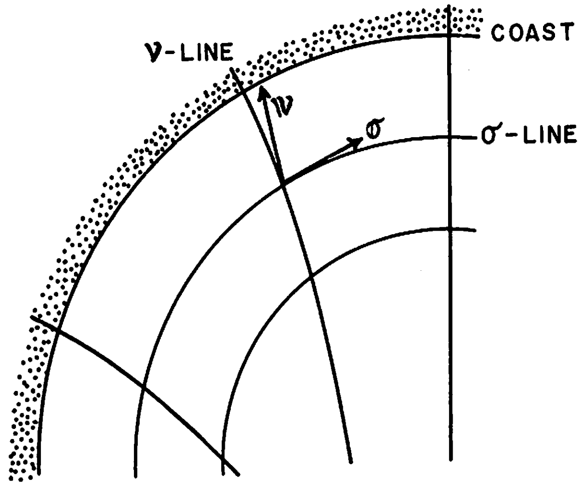


Figure 2. - Definition of bathystrophic coordinates.

The orthogonals to the σ-lines can be found from the differential equation

$$d\mathbf{S} \times \boldsymbol{\nu} = 0 \quad \text{i.e.} \quad \frac{dy}{dx} = \frac{\partial H}{\partial y} \Big/ \frac{\partial H}{\partial x} \tag{97}$$

where $d\mathbf{S}$ is a line element of the orthogonals. The solution $y=y(x,C)$ represents the family of ν-lines in the plane.

In this orthogonal coordinate system each vector can be written

$$\mathbf{A} = \boldsymbol{\sigma} A_\sigma + \boldsymbol{\nu} A_\nu \tag{98}$$

or, for the nabla

$$\nabla_h = \boldsymbol{\sigma}\frac{\partial}{\partial\sigma} + \boldsymbol{\nu}\frac{\partial}{\partial\nu} \tag{99}$$

The unit vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\nu}$ change their direction from point to point in the plane and are therefore dependent on the position vector. For example we get for the horizontal divergence of these vectors using (95), (96)

$$\nabla_h \cdot \boldsymbol{\sigma} = -\frac{1}{|\nabla_h H|^2}\,\mathbf{k}\cdot\nabla_h H \times \nabla_h(|\nabla_h H|) = \frac{\partial\alpha}{\partial\nu} = \frac{1}{R_\nu} \tag{100}$$

$$\nabla_h \cdot \boldsymbol{\nu} = \frac{1}{|\nabla_h H|^2}\nabla_h H \cdot \nabla_h(|\nabla_h H|) - \frac{1}{|\nabla_h H|}(\nabla_h^2 H) = -\frac{\partial\alpha}{\partial\sigma} = -\frac{1}{R_\sigma} \tag{101}$$

where the radii of curvature of the coordinate lines are R_σ and R_ν respectively and are defined by the rate of change of the angle α if one follows a σ-line or a ν-line in the direction of their unit vectors. These radii of curvature which play an important role in the following equations can be calculated from (100), (101) by means of simple differentiations of the bottom profile.

The following additional formulae for partial differentiations of the unit vectors are useful:

$$\left.\begin{array}{ll} \boldsymbol{\sigma}\cdot\nabla_h\boldsymbol{\sigma} = \dfrac{\partial\boldsymbol{\sigma}}{\partial\sigma} = \dfrac{1}{R_\sigma}\boldsymbol{\nu}; & \boldsymbol{\sigma}\cdot\nabla_h\boldsymbol{\nu} = -\dfrac{1}{R_\sigma}\boldsymbol{\sigma} \\ \boldsymbol{\nu}\cdot\nabla_h\boldsymbol{\sigma} = \dfrac{\partial\boldsymbol{\sigma}}{\partial\nu} = \dfrac{1}{R_\nu}\boldsymbol{\nu}; & \boldsymbol{\nu}\cdot\nabla_h\boldsymbol{\nu} = -\dfrac{1}{R_\nu}\boldsymbol{\sigma}. \end{array}\right\} \tag{102}$$

5.52 <u>Transformation of the basic equations to bathystrophic coordinates</u>

The basic equations of motion and continuity (55) and (41) can be transformed to bathystrophic coordinates by using the rules for differentiation of the unit vectors, i.e., (100), (101) and (102).

One gets for the equation of continuity (41)

$$\frac{\partial\bar{Q}}{\partial t} + \left(\boldsymbol{\sigma}\frac{\partial}{\partial\sigma} + \boldsymbol{\nu}\frac{\partial}{\partial\nu}\right) \cdot \left(\boldsymbol{\sigma}\bar{Q}\hat{V}_\sigma + \boldsymbol{\nu}\bar{Q}\hat{V}_\nu\right)$$

$$= \frac{\partial\bar{Q}}{\partial t} + \frac{\partial\bar{Q}\hat{V}_\sigma}{\partial\sigma} + \frac{\partial\bar{Q}\hat{V}_\nu}{\partial\nu} + \bar{Q}\hat{V}_\sigma \nabla_h \cdot \boldsymbol{\sigma} + \bar{Q}\hat{V}_\nu \nabla_h \cdot \boldsymbol{\nu} = \bar{\bar{P}}\text{-}\bar{\bar{E}}$$

or

$$\frac{\partial\bar{Q}}{\partial t} + \frac{\partial\bar{Q}\hat{V}_\sigma}{\partial\sigma} + \frac{\partial\bar{Q}\hat{V}_\nu}{\partial\nu} + \frac{\bar{Q}\hat{V}_\sigma}{R_\nu} - \frac{\bar{Q}\hat{V}_\nu}{R_\sigma} = \bar{\bar{P}}\text{-}\bar{\bar{E}}. \tag{103}$$

Scalarly multiplying the equation of motion (55) by $\boldsymbol{\sigma}$ and $\boldsymbol{\nu}$ and introducing (98), (99), one gets for the acceleration term the two components

$$\frac{\partial\bar{Q}\hat{V}_\sigma}{\partial t} + \nabla_h \cdot \left\{\bar{Q}\hat{\mathbf{V}}_h\hat{\mathbf{V}}_h\right\} \cdot \boldsymbol{\sigma} - f\bar{Q}\hat{V}_\nu$$

$$\frac{\partial\bar{Q}\hat{V}_\nu}{\partial t} + \nabla_h \cdot \left\{\bar{Q}\hat{\mathbf{V}}_h\hat{\mathbf{V}}_h\right\} \cdot \boldsymbol{\nu} + f\bar{Q}\hat{V}_\sigma \tag{104}$$

Using further the expressions for the components of the divergence of the dyadic product of two vectors $\mathbf{A}$ and $\mathbf{B}$

$$\nabla_h \cdot (\mathbf{AB}) \cdot \boldsymbol{\sigma} = \frac{\partial A_\sigma B_\sigma}{\partial\sigma} + \frac{\partial A_\nu B_\sigma}{\partial\nu} + \frac{A_\sigma B_\sigma - A_\nu B_\nu}{R_\nu} - \frac{A_\sigma B_\nu + A_\nu B_\sigma}{R_\sigma}$$

$$\nabla_h \cdot (\mathbf{AB}) \cdot \boldsymbol{\nu} = \frac{\partial A_\sigma B_\nu}{\partial\sigma} + \frac{\partial A_\nu B_\nu}{\partial\nu} + \frac{A_\sigma B_\sigma - A_\nu B_\nu}{R_\sigma} + \frac{A_\sigma B_\nu + A_\nu B_\sigma}{R_\nu} \tag{105}$$

in the acceleration components (104) and taking the σ- and ν-components of the friction term on the left side and of the terms on the right side of equation (55), one obtains the following system

(1) Equation of continuity:

$$\frac{\partial\bar{Q}}{\partial t} + \frac{\partial\bar{Q}\hat{V}_\sigma}{\partial\sigma} + \frac{\partial\bar{Q}\hat{V}_\nu}{\partial\nu} + \frac{\bar{Q}\hat{V}_\sigma}{R_\nu} - \frac{\bar{Q}\hat{V}_\nu}{R_\sigma} = \bar{\bar{P}}\text{-}\bar{\bar{E}} \tag{103}$$

(2) Components of the equations of motion:

$$\frac{\partial \bar{Q}\hat{V}_\sigma}{\partial t} + \frac{\partial}{\partial\sigma}(\bar{Q}\hat{V}_\sigma^2) + \frac{\partial}{\partial\nu}(\bar{Q}\hat{V}_\nu\hat{V}_\sigma) + \frac{\bar{Q}}{R_\nu}(\hat{V}_\sigma^2 - \hat{V}_\nu^2) - \frac{2\bar{Q}}{R_\sigma}\hat{V}_\nu\hat{V}_\sigma - f\bar{Q}\hat{V}_\nu$$

$$= \nabla_h\cdot(\bar{\bar{\mathcal{F}}}_h)\cdot\boldsymbol{\sigma} - g\bar{Q}\frac{\partial}{\partial\sigma}(\bar{\bar\eta}-\eta^{(T)}) - (\bar{\bar\eta}+H)\frac{\partial\bar{\bar p}^{(a)}}{\partial\sigma} + \bar{\tau}^{(a)}_{z,h}\cdot\boldsymbol\sigma - (\bar\tau_{z,h})_b\cdot\boldsymbol\sigma$$

$$- g\bar{\bar{\bar\rho}}\frac{\partial}{\partial\sigma}\left(\frac{\overline{\eta'^2}}{2}\right) - \frac{g}{2}\left\{(\bar{\bar\eta}+H)^2 + (\overline{\eta'^2})\right\}\frac{\partial\bar{\bar{\bar\rho}}}{\partial\sigma} - g\frac{\partial}{\partial\sigma}\left(\overline{\int_{-H}^{\bar\eta}dz\int_{z}^{\bar\eta}\bar\rho^{*}d\zeta}\right)$$

$$-\overline{\nabla_h\bar\eta'\cdot(\mathbf{i}\tau'^{(a)}_{x,h} + \mathbf{j}\tau'^{(a)}_{y,h})}\cdot\boldsymbol\sigma - g(\bar{\bar\eta}+H)\frac{\partial}{\partial\sigma}(\overline{\bar\rho'\bar\eta'}) - g\frac{\partial}{\partial\sigma}\left(\overline{\bar\rho'\frac{\bar\eta'^2}{2}}\right)$$

$$-\overline{\bar\eta'\frac{\partial\bar p'^{(a)}}{\partial\sigma}} - \nabla_h\bar{\bar\eta}\cdot(\mathbf{i}\bar\tau^{(a)}_{x,h} + \mathbf{j}\bar\tau^{(a)}_{y,h})\cdot\boldsymbol\sigma - \nabla_h H\cdot(\mathbf{i}(\bar\tau_{x,h})_b + \mathbf{j}(\bar\tau_{y,h})_b)$$

$$-\frac{\partial\overline{\mathcal{E}^{(i)}}}{\partial\sigma} + \overline{(\bar{\bar P}-\bar{\bar E})\hat{\mathbf{v}}_{h,s}\cdot\boldsymbol\sigma} = F_\sigma \tag{106}$$

$$\frac{\partial \bar{Q}\hat{V}_\nu}{\partial t} + \frac{\partial}{\partial\sigma}(\bar{Q}\hat{V}_\sigma\hat{V}_\nu) + \frac{\partial}{\partial\nu}(\bar{Q}\hat{V}_\nu^2) + \frac{\bar{Q}}{R_\sigma}(\hat{V}_\sigma^2 - \hat{V}_\nu^2) + \frac{2\bar{Q}}{R_\nu}\hat{V}_\sigma\hat{V}_\nu + f\bar{Q}\hat{V}_\sigma$$

$$= \nabla_h\cdot(\bar{\bar{\mathcal{F}}}_h)\cdot\boldsymbol{\nu} - g\bar{Q}\frac{\partial}{\partial\nu}(\bar{\bar\eta}-\eta^{(T)}) - (\bar{\bar\eta}+H)\frac{\partial\bar{\bar p}^{(a)}}{\partial\nu} + \bar{\tau}^{(a)}_{z,h}\cdot\boldsymbol\nu - (\bar\tau_{z,h})_b\cdot\boldsymbol\nu$$

$$- g\bar{\bar{\bar\rho}}\frac{\partial}{\partial\nu}\left(\frac{\overline{\eta'^2}}{2}\right) - \frac{g}{2}\left\{(\bar{\bar\eta}+H)^2 + (\overline{\eta'^2})\right\}\frac{\partial\bar{\bar{\bar\rho}}}{\partial\nu} - g\frac{\partial}{\partial\nu}\left(\overline{\int_{-H}^{\bar\eta}dz\int_{z}^{\bar\eta}\bar\rho^{*}d\zeta}\right)$$

$$-\overline{\nabla_h\bar\eta'\cdot(\mathbf{i}\tau'^{(a)}_{x,h} + \mathbf{j}\tau'^{(a)}_{y,h})}\cdot\boldsymbol\nu - g(\bar{\bar\eta}+H)\frac{\partial}{\partial\nu}(\overline{\bar\rho'\bar\eta'}) - g\frac{\partial}{\partial\nu}\left(\overline{\bar\rho'\frac{\bar\eta'^2}{2}}\right)$$

$$-\overline{\bar\eta'\frac{\partial\bar p'^{(a)}}{\partial\nu}} - \nabla_h\bar{\bar\eta}\cdot(\mathbf{i}\bar\tau^{(a)}_{x,h} + \mathbf{j}\bar\tau^{(a)}_{y,h})\cdot\boldsymbol\nu - \nabla_h H\cdot(\mathbf{i}(\bar\tau_{x,h})_b + \mathbf{j}(\bar\tau_{y,h})_b)\cdot\boldsymbol\nu$$

$$-\overline{\mathcal{E}_b}Q\left|\nabla_h H\right| - \frac{\partial\overline{\mathcal{E}^{(i)}}}{\partial\nu} + \overline{(\bar{\bar P}-\bar{\bar E})\hat{\mathbf{v}}_{h,s}}\cdot\boldsymbol\nu = F_\nu \tag{107}$$

where the deviation of the bottom pressure from the hydrostatically calculated one enters only in the ν-component.

5.53 Bathystrophic motions

The purpose of transformations of the hydrodynamical equations is to find forms of these equations which are suitable for the treatment of special physical problems. For example, in transformations to fulfill given boundary conditions, one tries to make the boundary surface (or line) a coordinate surface (or line). This is exactly the case if one uses the bathystrophic coordinates where the coast line is a coordinate line, a σ-line, for which under normal equilibrium conditions the water transport normal to this line vanishes. But besides this, the dependence of the terms on the right side of equations (106) and (107) on the coordinates σ and ν is a quite different one. For example, the component of the wave set-up term in the direction normal to the isobaths is certainly much greater than in the tangential direction. Further, as already mentioned, the non-hydrostatic bottom pressure deviation vanishes in one component only in this bathystrophic system. For many applications also the density gradient has the direction normal to the isobaths and therefore the component of the density gradient tangential to the isobaths in (106) can be neglected. There are many properties of the sea depending mainly on the distance from the coast; i.e., depending mainly on ν and not on σ. For this reason the equations written with respect to bathystrophic coordinates may be useful with respect to reasonable approximations.

One of these approximations is the assumption of a bathystrophic motion "in the mean" which means that the component of the vertically and time averaged mass transport normal to the contour lines σ = const vanishes. There certainly can be a mass transport toward the coast in the upper layers but there must be present also a lower counter-current for compensation. If there were a sustained mass transport toward the shore across the isobaths in the vicinity of the coast, much larger land areas would be inundated than is usually the case. In our notations the bathystrophic approximation, which was used successfully by Freeman, Baer, and Jung [12], means

$$\mathbb{V}_h = \hat{V}_\sigma \boldsymbol{\sigma}; \quad \hat{V}_\nu = 0 \tag{108}$$

By introducing the mass transport vector

$$\mathbf{M}_h = \bar{Q}\hat{\mathbb{V}}_h \quad ; \quad M_\sigma = \bar{Q}\hat{V}_\sigma \tag{109}$$

the bathystrophic approximated equations are obtained from (103), (106) and (107) in the following form

$$\frac{\partial \bar{Q}}{\partial t} + \left(\frac{\partial}{\partial \sigma} + \frac{1}{R_\nu}\right) M_\sigma = \bar{\bar{P}} - \bar{\bar{E}} \tag{103'}$$

$$\frac{\partial M_\sigma}{\partial t} + \left(\frac{\partial}{\partial \sigma} + \frac{1}{R_\nu}\right) \frac{M_\sigma^2}{\bar{Q}} = F_\sigma \tag{110}$$

$$f M_\sigma + \frac{M_\sigma^2}{\bar{Q} R_\sigma} = F_\nu \tag{111}$$

where the abridged right hand sides of equations (106) and (107) have been used. This much simplified system with known radii of curvature is the basic system for the calculation of the so called bathystrophic storm tide [12].

The bathystrophic mass transport M_σ follows from (111) immediately if the component of the "force" $\mathbb{F}_h^\nu$ in the direction ν is known:

$$M_\sigma = -\frac{1}{2} f\bar{Q}R_\sigma \left\{ 1 \pm \sqrt{1 + \frac{4F_\nu}{f^2\bar{Q}R_\sigma}} \right\} \tag{112}$$

or

$$\hat{V}_\sigma = -\frac{1}{2} fR_\sigma \left\{ 1 \pm \sqrt{1 + \frac{4F_\nu}{f^2\bar{Q}R_\sigma}} \right\} \tag{113}$$

The analogy to the geostrophic-cyclostrophic wind in meteorology is obvious; however, here the two signs of the square root together with the two possible signs of the radius of curvature lead to four possible solutions.

Since $\bar{Q}$ is always greater than zero, the sign and magnitude of F_ν and R_σ determine the direction and the magnitude of the bathystrophic mean volume (or mass) transport vector.

For "cyclonicly" curved isobaths, i.e., $R_\sigma > 0$, the two solutions of (113) are

$$\hat{V}^{(c)}_{\sigma,1} = -\frac{fR_\sigma}{2} \left\{ 1 + \sqrt{1 + \frac{4F_\nu}{f^2\bar{Q}R_\sigma}} \right\} \tag{114}$$

$$\hat{V}^{(c)}_{\sigma,2} = -\frac{fR_\sigma}{2} \left\{ 1 - \sqrt{1 + \frac{4F_\nu}{f^2\bar{Q}R_\sigma}} \right\} \tag{115}$$

$$\frac{F_\nu}{f\,\bar{Q}} \geqq -\frac{fR_\sigma}{4} \tag{116}$$

with the limit value

$$\hat{V}^{(c)}_{\sigma,1} = \hat{V}^{(c)}_{\sigma,2} = -\frac{fR_\sigma}{2} \tag{117}$$

for the equal sign in (116). This means an "anticyclonic" current in this "cyclonicly" curved family of isobaths for both solutions.

The case of straight isobaths is obtained from (111) and given by

$$\hat{V}^{(s)}_\sigma = \frac{F_\nu}{f\,\bar{Q}} \tag{118}$$

For "anticyclonicly" curved isobaths, i.e., $R_\sigma < 0$, the solutions are

$$\hat{V}^{(ac)}_{\sigma,1} = \frac{f|R_\sigma|}{2}\left\{1 + \sqrt{1 - \frac{4F_\nu}{f^2\overline{Q}|R_\sigma|}}\right\} \tag{119}$$

$$\hat{V}^{(ac)}_{\sigma,2} = \frac{f|R_\sigma|}{2}\left\{1 - \sqrt{1 - \frac{4F_\nu}{f^2\overline{Q}|R_\sigma|}}\right\} \tag{120}$$

where

$$\frac{F_\nu}{f\,\overline{Q}} \leqq \frac{f|R_\sigma|}{4} \tag{121}$$

with the limit for the case of the equal sign

$$\hat{V}^{(ac)}_{\sigma,1} = \hat{V}^{(ac)}_{\sigma,2} = \frac{f|R_\sigma|}{2} \tag{122}$$

Introducing the new variables X and Y defined by

$$X = \frac{2\hat{V}_\sigma}{f|R_\sigma|} \qquad Y = \frac{4F_\nu}{f^2\overline{Q}|R_\sigma|} \tag{123}$$

into (114), (115) and (119), (120), one gets for purposes of graphical representation

$$Y = X\,(2 + X) \; : \text{ for } R_\sigma > 0, \text{ i.e., cyclonicly curved isobaths} \tag{124}$$

$$Y = X\,(2 - X) \; : \text{ for } R_\sigma < 0, \text{ i.e., anticyclonicly curved isobaths} \tag{125}$$

We shall call the dashed parts of the curves in figure 3 and the corresponding dashed vectors in figure 4 anomalous solutions. These may be realized for bathystrophic currents on the shelf of the continents under certain conditions.

In the case of the geostrophic-cyclostrophic wind in meteorology the "force" F_ν corresponds to the pressure gradient pointing toward lower pressure and therefore in the atmosphere $F_\nu \sim -\frac{\partial p}{\partial \nu} > 0$ and only the upper part of figure 3 is valid for atmospheric motions. Then the dashed part of the curve for anticyclonic curvature is the anomalous solution for anticyclonic atmospheric motions. According to Alaka [1,2] this plays a role in hurricane dynamics. The dashed part of the curve for cyclonic curvature in the upper part of figure 3 corresponds to the antibaric anticyclonic motion in the atmosphere (Holmboe et al. [18]).

In the present case of oceanographic bathystrophic motions the sign of the "force" component F_ν normal to the isobaths cannot be determined in such an easy way as is done in connection with the geostrophic-cyclostrophic wind for a frictionless atmosphere but this sign is determined by the sum of all terms on the right hand side of equation (107). Therefore even the negative sign for F_ν may appear and consequently more kinds of bathystrophic motions

$$f M_\sigma + \frac{M_\sigma^2}{Q R_\sigma} = F_\nu$$

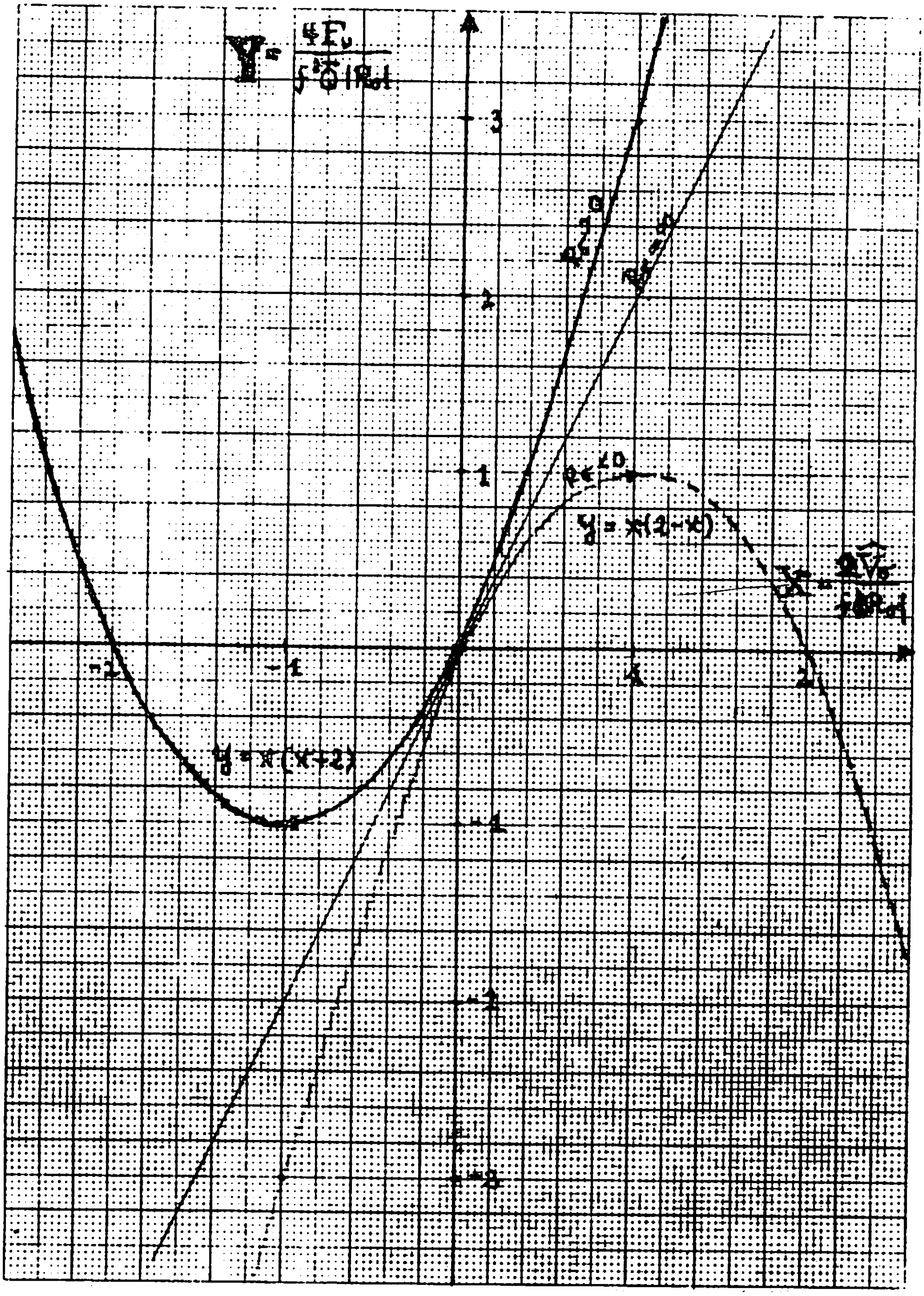


Figure 3. - Representation of the different solutions of the bathystrophic-current equation (111).

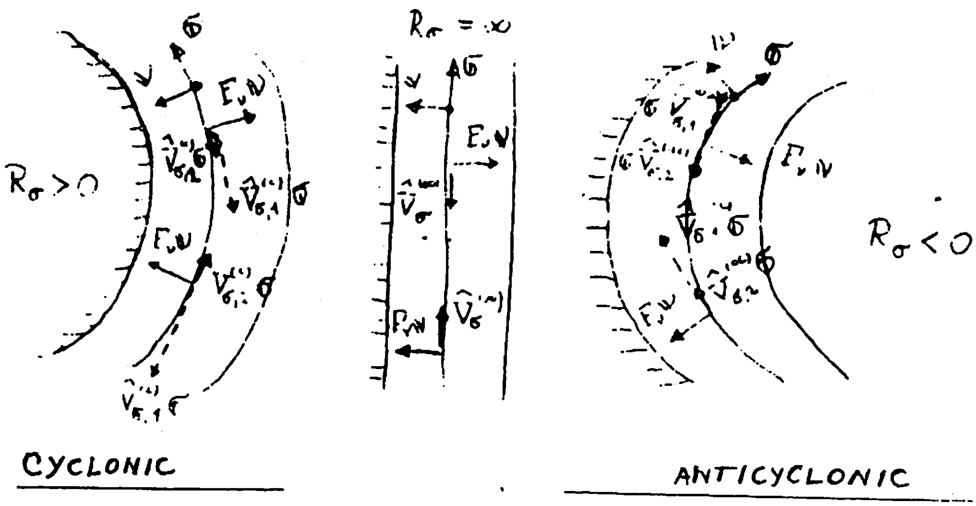


Figure 4. - Possible directions of the bathystrophic current for given type of curvature of the isobaths and given direction of "force."

are possible in the vicinity of the coast than are realized in the atmosphere for the simple model of the geostrophic-cyclostrophic wind.

In the case of cyclonic curvature of the isobaths, $R_\sigma > 0$, we have according to figures 3 and 4 for $F_\nu > 0$ the normal cyclonic current $\hat{V}^{(c)}_{\sigma,2}$ and the anomalous anticyclonic current $\hat{V}^{(c)}_{\sigma,1}$. For $F_\nu < 0$ we have only anticyclonic currents, a smaller normal solution, and a larger anomalous solution which, however, are bounded with the limit value (117).

In the case of anticyclonic curvature of the isobaths and for $F_\nu > 0$ the possible currents are anticyclonic and we have a smaller normal and a larger anomalous solution. These are bounded with the limit value given by (122). For $F_\nu < 0$ we have a smaller cyclonic current and a larger anticyclonic current.

The simplest expression for F_ν is obtained from (107) by neglecting all but one term; i.e.,

$$F_\nu = - g \, \overline{Q} \, \frac{\partial \bar{\bar{\eta}}}{\partial \nu}$$

Therefore for $\frac{\partial \bar{\bar{\eta}}}{\partial \nu} > 0$, $F_\nu < 0$, the currents have the coast on the right hand side (except the anomalous current for anticyclonic curvature of the isobaths). For $\frac{\partial \bar{\bar{\eta}}}{\partial \nu} < 0$, $F_\nu > 0$, the currents have the coast on the left hand side (except the anomalous current for cyclonic curvature of the isobaths).

When, however, all "forces" on the right hand side of (107) are present, such simple statements cannot be made, but the inequalities (116), (121) for given curvatures of the isobaths remain valid and can be used for the calculation of limits for the so called "set-up" in cases in which the curvature of the isobaths plays a significant role. If, however, any information about the direction and the magnitude of the bathystrophic current is available, even under hurricane conditions, the complete bathystrophic-current equation (111) can be used for an estimation of the set-up. If this is not the case, the complete system of equations for the bathystrophic mass transport, i.e., (110), (111) and (103') must be used in order to eliminate the bathystrophic mass transport and to obtain equations for the remaining variables. This was done, in simplified form by Freeman et al.[12]. The general non-linear problem, however, is a very complicated one, even if the bathystrophic mass transport can be eliminated with the aid of (112).

5.54 Elimination of the bathystrophic mass transport from the complete bathystrophic system of equations and the bathystrophic wave equation

The equation (111) for the bathystrophic mass transport allows the elimination of M_σ from the system (110), (111), and (103') and the derivation of a wave equation for the mass of a vertical column even in the quite general non-linear case described by these equations.

Writing (111) in the form

$$\frac{M_\sigma^2}{\bar{Q}} = - f R_\sigma M_\sigma + R_\sigma F_\nu \tag{126}$$

and introducing this into (110) gives

$$\frac{\partial M_\sigma}{\partial t} - f \frac{\partial R_\sigma}{\partial \sigma} M_\sigma = f R_\sigma \left(\frac{\partial M_\sigma}{\partial \sigma} + \frac{M_\sigma}{R_\nu} \right) - R_\sigma \left(\frac{\partial F_\nu}{\partial \sigma} + \frac{F_\nu}{R_\nu} \right) + \left(F_\sigma - \frac{\partial R_\sigma}{\partial \sigma} F_\nu \right) \tag{127}$$

with the divergence of the bathystrophic mass transport on the right hand side. This can be expressed by means of the equation of continuity (103') leading to

$$\left(\frac{\partial}{\partial t} - f \frac{\partial R_\sigma}{\partial \sigma} \right) M_\sigma = -fR_\sigma \left(\frac{\partial \bar{Q}}{\partial t} - (\bar{\bar{P}} - \bar{\bar{E}}) \right) - R_\sigma \left(\frac{\partial F_\nu}{\partial \sigma} + \frac{F_\nu}{R_\nu} \right) + \left(F_\sigma - \frac{\partial R_\sigma}{\partial \sigma} F_\nu \right) \tag{128}$$

Applying the divergence-operation $\frac{\partial}{\partial \sigma} + \frac{1}{R_\nu}$ on (128) gives

$$\left(\frac{\partial}{\partial t} - f \frac{\partial R_\sigma}{\partial \sigma} \right) \left(\frac{\partial M_\sigma}{\partial \sigma} + \frac{M_\sigma}{R_\nu} \right) - f \frac{\partial^2 R_\sigma}{\partial \sigma^2} M_\sigma = -\left(\frac{\partial}{\partial t} - f \frac{\partial R_\sigma}{\partial \sigma} \right) \left(\frac{\partial \bar{Q}}{\partial t} - (\bar{\bar{P}} - \bar{\bar{E}}) \right) - f \frac{\partial^2 R_\sigma}{\partial \sigma^2} M_\sigma$$

$$= \left(\frac{\partial}{\partial \sigma} + \frac{1}{R_\nu} \right) \left\{ -fR_\sigma \left(\frac{\partial \bar{Q}}{\partial t} - (\bar{\bar{P}} - \bar{\bar{E}}) \right) - R_\sigma \left(\frac{\partial F_\nu}{\partial \sigma} + \frac{F_\nu}{R_\nu} \right) + \left(F_\sigma - \frac{\partial R_\sigma}{\partial \sigma} F_\nu \right) \right\}$$

From this it follows that if $\frac{\partial^2 R_\sigma}{\partial\sigma^2} \neq 0$, a second expression for the bathystrophic mass transport besides (112) is given by

$$M_\sigma = -\frac{1}{f\,\frac{\partial^2 R_\sigma}{\partial\sigma^2}}\left[\left(\frac{\partial}{\partial t} - f\,\frac{\partial R_\sigma}{\partial\sigma}\right)\left(\frac{\partial\bar{Q}}{\partial t} - (\bar{\bar{P}}-\bar{\bar{E}})\right) + \left(\frac{\partial}{\partial\sigma} + \frac{1}{R_\nu}\right)\left\{-fR_\sigma\left(\frac{\partial\bar{Q}}{\partial t} - (\bar{\bar{P}}-\bar{\bar{E}})\right) - R_\sigma\left(\frac{\partial F_\nu}{\partial\sigma} + \frac{F_\nu}{R_\nu}\right) + \left(F_\sigma - \frac{\partial R_\sigma}{\partial\sigma}F_\nu\right)\right\}\right] \tag{129}$$

This expression for the bathystrophic mass transport contains all "force" components, the configuration of the isobaths, and the change of the mass of a vertical column with time.

The stationary bathystrophic mass transport for curved isobaths follows from (128) as

$$M_\sigma = -\frac{1}{f\,\frac{\partial R_\sigma}{\partial\sigma}}\left\{f\,R_\sigma\,(\bar{\bar{P}}-\bar{\bar{E}}) - R_\sigma\left(\frac{\partial F_\nu}{\partial\sigma} + \frac{F_\nu}{R_\nu}\right) + \left(F_\sigma - \frac{\partial R_\sigma}{\partial\sigma}F_\nu\right)\right\} \tag{130}$$

and the limit case of straight isobaths is described by

$$M_\sigma = \frac{1}{f}F_\nu \tag{131}$$

In these expressions for the bathystrophic mass transport the equation of continuity has been used which makes these expressions more general than is equation (112). For example, the following derivation of a wave equation is possible only by using (129) and can hardly be done with the use of (112).

Applying the operator $\frac{\partial}{\partial t} - f\,\frac{\partial R_\sigma}{\partial\sigma}$ on (129) and introducing (128) gives the following "wave" equation

$$\left[\left(\frac{\partial}{\partial t} - f\,\frac{\partial R_\sigma}{\partial\sigma}\right)^2 - f\left(\frac{\partial}{\partial t} - f\,\frac{\partial R_\sigma}{\partial\sigma}\right)\left(R_\sigma\left(\frac{\partial}{\partial\sigma} + \frac{1}{R_\nu}\right) + \frac{\partial R_\sigma}{\partial\sigma}\right) - f^2R_\sigma\,\frac{\partial^2 R_\sigma}{\partial\sigma^2}\right]\left(\frac{\partial\bar{Q}}{\partial t} - (\bar{\bar{P}}-\bar{\bar{E}})\right)$$

$$= -\left[\left(\frac{\partial}{\partial t} - f\,\frac{\partial R_\sigma}{\partial\sigma}\right)\left(\frac{\partial}{\partial\sigma} + \frac{1}{R_\nu}\right) + f\,\frac{\partial^2 R_\sigma}{\partial\sigma^2}\right]\left(F_\sigma - \frac{\partial R_\sigma}{\partial\sigma}F_\nu - R_\sigma\left(\frac{\partial F_\nu}{\partial\sigma} + \frac{F_\nu}{R_\nu}\right)\right) \tag{132}$$

or

$$D_1\,\frac{\partial\bar{Q}}{\partial t} = -D_2\left(F_\sigma - \frac{\partial R_\sigma}{\partial\sigma}F_\nu - R_\sigma\left(\frac{\partial F_\nu}{\partial\sigma} + \frac{F_\nu}{R_\nu}\right)\right) + D_1\,(\bar{\bar{P}}-\bar{\bar{E}}) \tag{133}$$

where

$$D_1 = \frac{\partial^2}{\partial t^2} - f\,R_\sigma \frac{\partial^2}{\partial t\partial\sigma} - f\left(3\frac{\partial R_\sigma}{\partial\sigma} + \frac{R_\sigma}{R_\nu}\right)\frac{\partial}{\partial t} + f^2 R_\sigma \frac{\partial R_\sigma}{\partial\sigma}\frac{\partial}{\partial\sigma} + f^2 \frac{\partial R_\sigma}{\partial\sigma}\frac{R_\sigma}{R_\nu} + 2f^2\left(\frac{\partial R_\sigma}{\partial\sigma}\right)^2$$

$$- f^2 R_\sigma \frac{\partial^2 R_\sigma}{\partial\sigma^2} \qquad (134)$$

$$D_2 = \frac{\partial^2}{\partial t\partial\sigma} + \frac{1}{R_\nu}\frac{\partial}{\partial t} - f\,\frac{\partial R_\sigma}{\partial\sigma}\frac{\partial}{\partial\sigma} + f\left(\frac{\partial^2 R_\sigma}{\partial\sigma^2} - \frac{1}{R_\nu}\frac{\partial R_\sigma}{\partial\sigma}\right) \qquad (135)$$

The corresponding equation for the steady state follows from (132) in the form

$$\left(\frac{\partial}{\partial\sigma} + \frac{1}{R_\nu} - \frac{\partial^2 R_\sigma}{\partial\sigma^2}\Big/\frac{\partial R_\sigma}{\partial\sigma}\right)\left(F_\sigma - \frac{\partial R_\sigma}{\partial\sigma}F_\nu - R_\sigma\left(\frac{\partial F_\nu}{\partial\sigma} + \frac{F_\nu}{R_\nu}\right)\right)$$

$$= -f\left(R_\sigma\left(\frac{\partial}{\partial\sigma} + \frac{1}{R_\nu}\right) + 2\frac{\partial R_\sigma}{\partial\sigma} - R_\sigma\frac{\partial^2 R_\sigma}{\partial\sigma^2}\Big/\frac{\partial R_\sigma}{\partial\sigma}\right)(\bar{\bar{P}}-\bar{\bar{E}}) \qquad (136)$$

This equation seems to have more physical meaning than (132) for the assumption (108) of pure bathystrophic motion means that streamlines and trajectories of the mean motion coincide with the isobaths and therefore the whole motion has to be steady state. Only in the case of a bathystrophic approximation, where $\hat{V}_\nu$ is considered to be small compared with other terms in the system of equations, the wave equation (132) is valuable as an approximate equation for motion which does not follow the isobaths exactly.

The steady state set-up in the case of curved isobaths disregarding the precipitation minus evaporation effect can be calculated from (136) by putting the right side equal to zero. The result is a second order partial differential equation in the variables σ and ν. This follows after introduction of the right hand sides of equations (106) and (107) into (136).

In the open sea the mean motion certainly is not bathystrophic and therefore the complete equations (103), (106), (107) have to be used, i.e.,

$$\frac{\partial\bar{Q}}{\partial t} + \left(\frac{\partial}{\partial\sigma} + \frac{1}{R_\nu}\right)M_\sigma + \left(\frac{\partial}{\partial\nu} - \frac{1}{R_\sigma}\right)M_\nu = \bar{\bar{P}}-\bar{\bar{E}} \qquad (103')$$

$$\frac{\partial M_\sigma}{\partial t} + \left(\frac{\partial}{\partial\sigma} + \frac{1}{R_\nu}\right)\frac{M_\sigma^2}{\bar{Q}} + \left(\frac{\partial}{\partial\nu} - \frac{1}{R_\sigma}\right)\frac{M_\sigma M_\nu}{\bar{Q}} - \frac{M_\sigma M_\nu}{\bar{Q}\,R_\sigma} - \frac{M_\nu^2}{\bar{Q}\,R_\nu} - f\,M_\nu = F_\sigma \qquad (106')$$

$$\frac{\partial M_\nu}{\partial t} + \left(\frac{\partial}{\partial\sigma} + \frac{1}{R_\nu}\right)\frac{M_\sigma M_\nu}{\bar{Q}} + \left(\frac{\partial}{\partial\nu} - \frac{1}{R_\sigma}\right)\frac{M_\nu^2}{\bar{Q}} + \frac{M_\sigma M_\nu}{\bar{Q}\,R_\nu} + \frac{M_\sigma^2}{\bar{Q}R_\sigma} + f\,M_\sigma = F_\nu \qquad (107')$$

where, however, the component M_ν decreases towards the coast and vanishes at the coast itself under certain conditions.

5.55 Separation of a basic current in bathystrophic coordinates in the case of a homogeneous ocean

For the sake of completeness we add to this chapter the transformation of the equations for a homogeneous ocean from which a basic current has been subtracted, i.e., the two systems (69), (70) and (77), (78). This transformation can be done easily with the aid of the properties given in 5.51 leading almost immediately to the equations for the pure storm surge properties in bathystrophic coordinates:

$$\frac{\partial \bar{\bar{\eta}}^{(*)}}{\partial t} + \left(\frac{\partial}{\partial \sigma} + \frac{1}{R_\nu}\right)\left\{(\bar{\bar{\eta}}^{(o)}+H)\hat{V}_\sigma^{(*)}+\bar{\bar{\eta}}^{(*)}\hat{V}_\sigma\right\} + \left(\frac{\partial}{\partial \nu} - \frac{1}{R_\sigma}\right)\left\{(\bar{\bar{\eta}}^{(o)}+H)\hat{V}_\nu^{(*)}+\bar{\bar{\eta}}^{(*)}\hat{V}_\nu\right\} = \frac{\bar{\bar{P}}-\bar{\bar{E}}}{\rho}$$

$$\frac{\partial \hat{V}_\sigma^{(*)}}{\partial t} + \hat{V}_\sigma \frac{\partial \hat{V}_\sigma^{(*)}}{\partial \sigma} + \hat{V}_\nu \frac{\partial \hat{V}_\sigma^{(*)}}{\partial \nu} + \hat{V}_\sigma^{(*)} \frac{\partial \hat{V}_\sigma^{(o)}}{\partial \sigma} + \hat{V}_\nu^{(*)} \frac{\partial \hat{V}_\sigma^{(o)}}{\partial \nu} - \qquad (69')$$

$$- \frac{1}{R_\sigma}\left(\hat{V}_\sigma \hat{V}_\nu^{(*)} + \hat{V}_\sigma^{(*)}\hat{V}_\nu^{(o)}\right) - \frac{1}{R_\nu}\left(\hat{V}_\nu \hat{V}_\nu^{(*)} + \hat{V}_\nu^{(*)}\hat{V}_\nu^{(o)}\right) -$$

$$- f\hat{V}_\nu^{(*)} + \frac{\bar{\bar{P}}-\bar{\bar{E}}}{\rho}\hat{V}_\sigma = - \frac{\partial}{\partial \sigma}\left(\bar{\bar{\eta}}^{(*)} - \bar{\eta}^{(a)}\right) + \sigma\text{-component of}$$

the additional terms on the right hand side of (70).

$$\frac{\partial \hat{V}_\nu^{(*)}}{\partial t} + \hat{V}_\sigma \frac{\partial \hat{V}_\nu^{(*)}}{\partial \sigma} + \hat{V}_\nu \frac{\partial \hat{V}_\nu^{(*)}}{\partial \nu} + \hat{V}_\sigma^{(*)} \frac{\partial \hat{V}_\nu^{(o)}}{\partial \sigma} + \hat{V}_\nu^{(*)} \frac{\partial \hat{V}_\nu^{(o)}}{\partial \nu} +$$

$$+ \frac{1}{R_\sigma}\left(\hat{V}_\sigma \hat{V}_\sigma^{(*)} + \hat{V}_\sigma^{(*)}\hat{V}_\sigma^{(o)}\right) + \frac{1}{R_\nu}\left(\hat{V}_\nu \hat{V}_\sigma^{(*)} + \hat{V}_\nu^{(*)}\hat{V}_\sigma^{(o)}\right) + \qquad (70')$$

$$+ f\hat{V}_\sigma^{(*)} + \frac{\bar{\bar{P}}-\bar{\bar{E}}}{\rho}\hat{V}_\nu = - \frac{\partial}{\partial \nu}\left(\bar{\bar{\eta}}^{(*)} - \bar{\eta}^{(a)}\right) + \nu\text{-component of}$$

the additional terms on the right hand side of (70).

The second set of equations follows from (77), (78) in abridged form:

$$\frac{\partial \bar{\bar{\eta}}^{(*)}}{\partial t} + \left(\frac{\partial}{\partial \sigma} + \frac{1}{R_\nu}\right) S_\sigma^{(*)} + \left(\frac{\partial}{\partial \nu} - \frac{1}{R_\sigma}\right) S_\nu^{(*)} = \frac{\bar{\bar{P}}-\bar{\bar{E}}}{\rho}$$

$$(77')$$

$$\frac{\partial S_\sigma^{(*)}}{\partial t} + \nabla_h \cdot \left\{\ldots\right\} \cdot \sigma - f\, S_\nu^{(*)} = - g(\bar{\bar{\eta}}+H)\frac{\partial}{\partial \sigma}\left(\bar{\bar{\eta}}^{(*)} - \bar{\eta}^{(a)}\right) - g\bar{\bar{\eta}}^{(*)}\frac{\partial}{\partial \sigma}\left(\bar{\bar{\eta}}^{(o)} - \eta^{(T)}\right) +$$

$$+ \sigma\text{-component of further terms}$$

$$\frac{\partial S_\nu^{(*)}}{\partial t} + \nabla_h \cdot \left\{\ldots\right\} \cdot \nu + f\, S_\sigma^{(*)} = -g(\bar{\bar{\eta}}+H)\frac{\partial}{\partial \nu}\left(\bar{\bar{\eta}}^{(*)} - \bar{\eta}^{(a)}\right) - g\bar{\bar{\eta}}^{(*)}\frac{\partial}{\partial \nu}\left(\bar{\bar{\eta}}^{(o)} - \eta^{(T)}\right) +$$

$+ \nu$ -component of further terms, where the divergence terms on the left sides can be written explicitly with the aid of (105).

5.6 Time dependent coordinates

5.61 Natural coordinates

The reason for writing the basic equations (41) and (55) in this special form by introducing the variables $\overline{Q}$ and $\hat{\mathbb{V}}_h$ was to get equations analogous to the equations of two-dimensional gas dynamics. However, this leaves quite general forces on the right hand side of (55). This allows us to apply some standard techniques of transformation to general time-dependent coordinates and to derive some general theorems connected with the vorticity or the divergence of the mean volume transport vector $\hat{\mathbb{V}}_h$.

The introduction of so-called natural coordinates can be made analogous to the introduction of the bathystrophic coordinates. Writing the equations (41), (55) in abridged form for the mass transport vector

$$\frac{\partial\overline{Q}}{\partial t} + \nabla_h \cdot \mathbf{M}_h = \overline{\overline{P}}\text{-}\overline{\overline{E}} \tag{41'}$$

$$\frac{\partial\mathbf{M}_h}{\partial t} + \nabla_h \cdot \left\{ \frac{\mathbf{M}_h\mathbf{M}_h}{\overline{Q}} \right\} + f\mathbf{k}\times\mathbf{M}_h = \mathbb{F}^{\mathbf{v}}_h. \tag{55'}$$

where the force $\mathbb{F}^{\mathbf{v}}_h$ includes the lateral friction term. The natural coordinates are defined by the tangential vector $\mathbf{s}$ of a trajectory of the mean mass transport i.e.

$$\mathbf{M}_h = \mathbf{s}M \tag{137}$$

and the normal vector to the trajectories which is in the two-dimensional case also the normal vector to the streamlines.

$$\mathbf{n} = \mathbf{k}\times\mathbf{s} \tag{138}$$

Therefore streamlines and trajectories have the same tangential and normal vectors.

We call the streamlines the "s-lines" and the orthogonals of this system of lines the "n-lines." Then for each fixed moment the following relationships analogous to (100), (101) and (102) are derived in texts of differential geometry:

$$\nabla_h \cdot \mathbf{s} = \frac{1}{R_n} \quad ; \quad \nabla_h \cdot \mathbf{n} = -\frac{1}{R_s} \tag{139}$$

$$\left.\begin{aligned} \mathbf{s}\cdot\nabla_h\mathbf{s} &= \frac{\partial\mathbf{s}}{\partial s} = \frac{1}{R_s}\mathbf{n} \quad ; \quad \mathbf{s}\cdot\nabla_h\mathbf{n} = \frac{\partial\mathbf{n}}{\partial s} = -\frac{1}{R_s}\mathbf{s} \\ \mathbf{n}\cdot\nabla_h\mathbf{s} &= \frac{\partial\mathbf{s}}{\partial n} = \frac{1}{R_n}\mathbf{n} \quad ; \quad \mathbf{n}\cdot\nabla_h\mathbf{n} = \frac{\partial\mathbf{n}}{\partial n} = -\frac{1}{R_n}\mathbf{s} \end{aligned}\right\} \tag{140}$$

where the curvatures of the streamlines and their orthogonals are unknown functions of position and time in contrast to the case of bathystrophic coordinates.

Introducing the representation of any vector $\mathbf{A}$ and the nabla in natural coordinates i.e.

$$\mathbf{A} = \mathbf{s}A_s + \mathbf{n}A_n \quad ; \quad \nabla_h = \mathbf{s}\frac{\partial}{\partial s} + \mathbf{n}\frac{\partial}{\partial n} \quad . \tag{141}$$

into (41') and (55') together with (137) one obtains

$$\frac{\partial \bar{Q}}{\partial t} + \nabla_h \cdot \left| \mathbf{s}M \right| = \frac{\partial \bar{Q}}{\partial t} + \frac{\partial M}{\partial s} + \frac{M}{R_n} = \bar{\bar{P}} - \bar{\bar{E}}$$

$$\mathbf{s}\frac{\partial M}{\partial t} + M\frac{\partial \mathbf{s}}{\partial t} + \nabla_h \cdot \left| \frac{M^2}{\bar{Q}} \mathbf{s}\mathbf{s} \right| + fM\mathbf{n}$$

$$= \mathbf{s}\frac{\partial M}{\partial t} + M\frac{\partial \mathbf{s}}{\partial t} + \frac{\partial}{\partial s}\left\{\frac{M^2}{\bar{Q}}\right\}\mathbf{s} + \frac{M^2}{\bar{Q}}\left(\frac{1}{R_n}\mathbf{s} + \frac{1}{R_s}\mathbf{n}\right) + fM\mathbf{n} = \mathbf{s}F_s + \mathbf{n}F_n$$

Introducing the curvature $\frac{1}{r_s}$ of the trajectory according to Blaton's [3] equation

$$\frac{1}{r_s} = \frac{1}{R_s} + \frac{1}{\hat{V}}\,\mathbf{n} \cdot \frac{\partial \mathbf{s}}{\partial t} \tag{142}$$

we get the basic system in natural coordinates by taking the components of the preceding equations

$$\frac{\partial M}{\partial t} + \frac{\partial}{\partial s}\left\{\frac{M^2}{\bar{Q}}\right\} + \frac{M^2}{\bar{Q}\,R_n} = F_s \,, \tag{143}$$

$$fM + \frac{M^2}{\bar{Q}\,r_s} = F_n \,, \tag{144}$$

$$\frac{\partial \bar{Q}}{\partial t} + \frac{\partial M}{\partial s} + \frac{M}{R_n} = \bar{\bar{P}} - \bar{\bar{E}} \,. \tag{145}$$

In steady state is $r_s = R_s$ and the system becomes

$$\frac{\partial}{\partial s}\left\{\frac{M^2}{\bar{Q}}\right\} + \frac{M^2}{\bar{Q}\,R_n} = F_s \,, \qquad fM + \frac{M^2}{\bar{Q}\,R_s} = F_n \,, \qquad \frac{\partial M}{\partial s} + \frac{M}{R_n} = \bar{\bar{P}} - \bar{\bar{E}} \,. \tag{146}$$

Comparing (143), (144), (145) with (110), (111), (103') one can see that these systems are identical with the exception, that the curvatures of the isobaths have to be replaced by the corresponding curvatures of the streamlines and the trajectories and, further, that the system of unit vectors $\mathbf{s}$ and $\mathbf{n}$ is always orientated here in such a way that the tangential vector is the unit vector pointing in the direction of mass transport. However, as already mentioned, one essential difference exists between these two systems, in the bathystrophic system due to the assumption (108) of bathystrophic motion the curvatures R_σ and R_ν are known functions of position and can be calculated easily from the definitions (100) and (101). This is not the case for natural coordinates where the curvatures r_s and R_n can be obtained only by solving the whole system. Nevertheless, the equations (128), (129) and the wave equation have their formal counterparts in the following exact equations:

$$\frac{\partial M}{\partial t} - f\frac{\partial r_s}{\partial s} M = f r_s\left(\frac{\partial M}{\partial s} + \frac{M}{R_n}\right) - r_s\left(\frac{\partial F_n}{\partial s} + \frac{F_n}{R_n}\right) + \left(F_s - \frac{\partial r_s}{\partial s} F_n\right) \tag{147}$$

or

$$\frac{\partial M}{\partial t} - f\frac{\partial r_s}{\partial s} M = - f r_s\left(\frac{\partial \bar{Q}}{\partial t} - (\bar{\bar{P}} - \bar{\bar{E}})\right) - r_s\left(\frac{\partial F_n}{\partial s} + \frac{F_n}{R_n}\right) + \left(F_s - \frac{\partial r_s}{\partial s} F_n\right) \tag{148}$$

$$M = -\frac{1}{f\dfrac{\partial^2 r_s}{\partial s^2} - \dfrac{1}{R_n^2}\dfrac{\partial R_n}{\partial t}}\left[\left(\frac{\partial}{\partial t} - f\frac{\partial r_s}{\partial s}\right)\left(\frac{\partial \bar{Q}}{\partial t} - (\bar{\bar{P}} - \bar{\bar{E}})\right)\right.$$
$$\left. + \left(\frac{\partial}{\partial s} + \frac{1}{R_n}\right)\left| - f r_s\left(\frac{\partial \bar{Q}}{\partial t} - (\bar{\bar{P}} - \bar{\bar{E}})\right) - r_s\left(\frac{\partial F_n}{\partial s} + \frac{F_n}{R_n}\right) + \left(F_s - \frac{\partial r_s}{\partial s} F_n\right)\right|\right] \tag{149}$$

$$\left[\left(\frac{\partial}{\partial t} - f\frac{\partial r_s}{\partial s}\right)^2 - f\left(\frac{\partial}{\partial t} - f\frac{\partial r_s}{\partial s}\right)\left(r_s\left(\frac{\partial}{\partial s} + \frac{1}{R_n}\right) + \frac{\partial r_s}{\partial s}\right) - f r_s\left(f\frac{\partial^2 r_s}{\partial s^2} + \frac{\partial}{\partial t}\left(\frac{1}{R_n}\right)\right) - \right.$$
$$\left. - \frac{\partial}{\partial t}\left(\ln\left(f\frac{\partial^2 r_s}{\partial s^2} + \frac{\partial}{\partial t}\left(\frac{1}{R_n}\right)\right)\right)\left(\frac{\partial}{\partial t} - f\frac{\partial r_s}{\partial s} - f\left(r_s\left(\frac{\partial}{\partial s} + \frac{1}{R_n}\right) + \frac{\partial r_s}{\partial s}\right)\right)\right]\left(\frac{\partial \bar{Q}}{\partial t} - (\bar{\bar{P}} - \bar{\bar{E}})\right)$$
$$= -\left[\left(\frac{\partial}{\partial t} - f\frac{\partial r_s}{\partial s}\right)\left(\frac{\partial}{\partial s} + \frac{1}{R_n}\right) + \left(f\frac{\partial^2 r_s}{\partial s^2} + \frac{\partial}{\partial t}\left(\frac{1}{R_n}\right)\right) + \frac{\partial}{\partial t}\left(\ln\left(f\frac{\partial^2 r_s}{\partial s^2} + \frac{\partial}{\partial t}\left(\frac{1}{R_n}\right)\right)\right)\left(\frac{\partial}{\partial s} + \frac{1}{R_n}\right)\right]$$
$$\left(F_s - \frac{\partial r_s}{\partial s} F_n - r_s\left(\frac{\partial F_n}{\partial s} + \frac{F_n}{R_n}\right)\right) \tag{150}$$

The differential operators in equation (150) are not interchangeable, for the curvatures here are also functions of time.

The steady state equations are obtained from these equations in the form

$$M = -\frac{1}{f\,\frac{\partial R_s}{\partial s}}\left\{ fR_s(\bar{\bar{P}}-\bar{\bar{E}}) - R_s\left(\frac{\partial F_n}{\partial s} + \frac{F_n}{R_n}\right) + \left(F_s - \frac{\partial R_s}{\partial s}F_n\right)\right\} \tag{151}$$

$$\left(\frac{\partial}{\partial s} + \frac{1}{R_n} - \frac{\partial^2 R_s}{\partial s^2}\Big/\frac{\partial R_s}{\partial s}\right)\left(F_s - \frac{\partial R_s}{\partial s}F_n - R_s\left(\frac{\partial F_n}{\partial s} + \frac{F_n}{R_n}\right)\right)$$

$$= -f\left(R_s\left(\frac{\partial}{\partial s} + \frac{1}{R_n}\right) + 2\frac{\partial R_s}{\partial s} - R_s\frac{\partial^2 R_s}{\partial s^2}\Big/\frac{\partial R_s}{\partial s}\right)(\bar{\bar{P}}-\bar{\bar{E}}). \tag{152}$$

5.62 Coordinates fixed in the center of the moving hurricane

For some purposes it is convenient to transform the basic equations (41), (55) to a coordinate system which is fixed in the center of the moving hurricane. For example, in this system some of the acting forces can be considered almost independent of time; e.g., the atmospheric pressure field and also the field of the wind stresses on the surface of the sea. By this transformation the problem becomes a steady state one if the bottom of the sea is flat and the boundaries are sufficiently distant from the hurricane center, and if further, no horizontal density gradients and no basic current exist in the free ocean. If, however, the bottom is not flat in the moving system, the bottom topography becomes a function of time even if the bottom is at rest with respect to the earth. The coast line (in this moving coordinate system) is moving toward the hurricane, if the hurricane is approaching the coast.

We consider a moving rigid coordinate system x', y' fixed in the center of the hurricane which rotates about the axis of the hurricane with angular velocity Ω and which has also the translation velocity of the center $\mathbf{C}_\zeta$. Then a point P of the moving system has the velocity $\mathbf{C}$ relative to the earth

$$\mathbf{C} = \mathbf{C}_\zeta + \Omega\,\mathbf{k}\times\mathbf{r} \tag{153}$$

In terms of a coordinate transformation we can also write

$$\mathbf{R} = \mathbf{R}_\zeta(t) + \Phi(t)\cdot\mathbf{r}$$

$$x = x_\zeta(t) + a_{xx}(t)x' + a_{xy}(t)\,y'$$

$$y = y_\zeta(t) + a_{yx}(t)x' + a_{yy}(t)\,y' \tag{154}$$

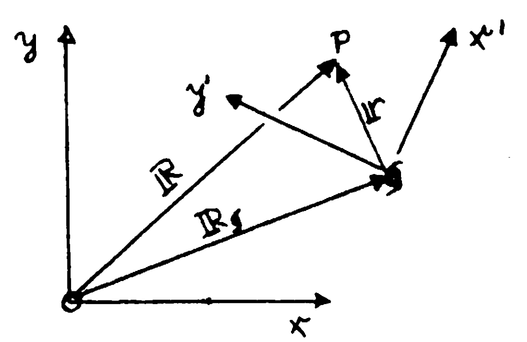


Figure 5. - Resting and moving frames.

where

$$\Phi = \begin{pmatrix} a_{xx} & a_{xy} \\ a_{yx} & a_{yy} \end{pmatrix} \quad \text{with} \quad \dot{\Phi} = \Omega(\mathbf{ji}-\mathbf{ij})$$

is an orthogonal tensor describing the rotation of the x', y'-system about the center of the hurricane.

Therefore the bottom topography becomes time dependent $H(x,y) = H(x',y',t)$ and further for a fixed point in the relative system the depth of the water below still water changes with time according to

$$\frac{\partial^{(rel)}H}{\partial t} = \mathbf{C} \cdot \nabla_h H \tag{155}$$

In order to transform the system (41), (55) to the moving system we write (41') and (55') in terms of the mean velocity $\hat{\mathbf{V}}_h$, i.e.

$$\frac{\hat{d}_h \bar{Q}}{dt} + \bar{Q}\,\nabla_h \cdot \hat{\mathbf{V}}_h = \bar{\bar{P}}-\bar{\bar{E}} \tag{41''}$$

$$\frac{\hat{d}_h \hat{\mathbf{V}}_h}{dt} + f\mathbf{k}\times\hat{\mathbf{V}}_h + \frac{\bar{\bar{P}}-\bar{\bar{E}}}{\bar{Q}}\hat{\mathbf{V}}_h = \frac{\mathbf{F}_h^{\nu}}{\bar{Q}} \tag{55''}$$

and write further for the mean velocity $\hat{\mathbf{V}}_h = \dfrac{\hat{d}_h \mathbf{R}}{dt}$.

The transformations of the velocity and the acceleration are well-known (Truesdell and Toupin [34]) and are given for our present purposes by

$$\hat{\mathbf{V}}_h = \hat{\mathbf{V}}_h^{(rel)} + \mathbf{C} = \hat{\mathbf{V}}_h^{(rel)} + \mathbf{C}_\zeta + \Omega\,(\mathbf{k}\times\mathbf{r}) \tag{156}$$

$$\frac{\hat{d}_h \hat{\mathbf{V}}_h}{dt} = \frac{\hat{d}_h^{(rel)}\hat{\mathbf{V}}_h^{(rel)}}{dt} + 2\Omega(\mathbf{k}\times\hat{\mathbf{V}}_h^{(rel)}) + \dot{\mathbf{C}}_\zeta + \dot{\Omega}(\mathbf{k}\times\mathbf{r}) - \Omega^2\mathbf{r}$$

$$= \frac{\partial^{(rel)}\hat{\mathbf{V}}_h^{(rel)}}{\partial t} + \hat{\mathbf{V}}_h^{(rel)} \cdot \nabla_h \hat{\mathbf{V}}_h^{(rel)} + 2\Omega(\mathbf{k}\times\hat{\mathbf{V}}_h^{(rel)})$$

$$+ \dot{\mathbf{C}}_\zeta + \dot{\Omega}(\mathbf{k}\times\mathbf{r}) - \Omega^2\mathbf{r} \tag{157}$$

where the dot means time derivation of functions which are only dependent on time as $\mathbf{C}_\zeta$ and Ω.

Considering all quantities in (41") and (55") as functions of x' and y', where the nabla appearing in $\mathbb{F}_h$ represents a nabla with derivatives with respect to the primed variables, the transformation is performed if one introduces (156) and (157) into (41") and (55"), observing that

$$\frac{\hat{d}_h \bar{Q}}{dt} = \frac{\hat{d}_h^{(rel)} \bar{Q}}{dt} \; ; \; \nabla_h \cdot \hat{\mathbb{V}}_h = \nabla_h \cdot \hat{\mathbb{V}}_h^{(rel)} \; ; \; \nabla_h \cdot \mathbf{C} = 0 \; .$$

One gets

$$\frac{\hat{d}_h^{(rel)} \bar{Q}}{dt} + \bar{Q} \, \nabla_h \cdot \hat{\mathbb{V}}_h^{(rel)} = \bar{\bar{P}} - \bar{\bar{E}} \tag{158}$$

$$\frac{\hat{d}_h^{(rel)} \hat{\mathbb{V}}_h^{(rel)}}{dt} + (f + 2\Omega)\,(\mathbf{k} \times \hat{\mathbb{V}}_h^{(rel)}) + \frac{\bar{\bar{P}} - \bar{\bar{E}}}{\bar{Q}} \hat{\mathbb{V}}_h^{(rel)} = \frac{\mathbb{F}_h}{\bar{Q}} \tag{159}$$

$$- \left\{ \dot{\mathbf{C}}_{\mathfrak{g}} + f \mathbf{k} \times \mathbf{C}_{\mathfrak{g}} + \frac{\bar{\bar{P}} - \bar{\bar{E}}}{\bar{Q}} \mathbf{C}_{\mathfrak{g}} + (\dot{\Omega} + \frac{\bar{\bar{P}} - \bar{\bar{E}}}{\bar{Q}} \Omega)(\mathbf{k} \times \mathbf{r}) - \Omega(\Omega + f)\,\mathbf{r} \right\}$$

or, in terms of the relative mass transport vector $\mathbf{M}_h^{(rel)}$

$$\frac{\partial^{(rel)} \bar{Q}}{\partial t} + \nabla_h \cdot \mathbf{M}_h^{(rel)} = \bar{\bar{P}} - \bar{\bar{E}} \tag{160}$$

$$\frac{\partial^{(rel)} \mathbf{M}_h^{(rel)}}{\partial t} + \nabla_h \cdot \left\{ \frac{\mathbf{M}_h^{(rel)} \, \mathbf{M}_h^{(rel)}}{\bar{Q}} \right\} + (f + 2\Omega)(\mathbf{k} \times \mathbf{M}_h^{(rel)}) = \mathbb{F}_h$$

$$- \bar{Q} \left\{ \dot{\mathbf{C}}_{\mathfrak{g}} + f \mathbf{k} \times \mathbf{C}_{\mathfrak{g}} + \frac{\bar{\bar{P}} - \bar{\bar{E}}}{\bar{Q}} \mathbf{C}_{\mathfrak{g}} + (\dot{\Omega} + \frac{\bar{\bar{P}} - \bar{\bar{E}}}{\bar{Q}} \Omega)(\mathbf{k} \times \mathbf{r}) - \Omega(\Omega + f)\,\mathbf{r} \right\} \tag{161}$$

This system differs from the original one (41'), (55') only by a modified Coriolis parameter and an additional force on the right side of (161) resulting from the acceleration and the velocity of the moving frame. Here, however, the apparent depth H of the bottom below still water level necessarily must be considered as a function of time and therefore it was convenient to carry out the whole calculations with time dependent H from the very beginning of these considerations. However this fact of a time dependent bottom may be more disadvantageous than having a time dependent atmospheric field of pressure and wind stress in the basic equations with respect to fixed coordinates.

6. FURTHER GENERAL EQUATIONS

The following considerations deal with some general equations which can be derived from the basic vertically integrated equations in analogy to some standard techniques used in meteorology. These special techniques can be applied immediately to the vertically integrated equations in the form (41"), (55") for the mean velocity vector, $\hat{\mathbb{V}}_h$, which differ from the corresponding equations for two-dimensional atmospheric motions only on the right hand sides which have a quite different physical meaning here than in the case in atmospheric dynamics.

Such derived new equations are mainly the vorticity, divergence, and energy equations. These are used widely instead of the so-called "primitive" equations which are here given by (41"), (55"). These equations can also be derived, however, from the equations for the mass transport, but then the analogy to the equations for compressible fluids vanishes.

The right hand vector $\mathbb{F}_h / \bar{Q}$ in (41") can be expressed always as the gradients of two scalar functions ϕ_D and ϕ_R, i.e.

$$\frac{\mathbb{F}_h}{\bar{Q}} = - \nabla_h \phi_D - \mathbf{k} \times \nabla_h \phi_R \tag{162}$$

where the vorticity and the divergence of this vector are given by

$$\mathbf{k} \cdot \nabla_h \times \left(\frac{\mathbb{F}_h}{\bar{Q}} \right) = - \nabla_h^2 \phi_D ; \quad \nabla_h \cdot \left(\frac{\mathbb{F}_h}{\bar{Q}} \right) = - \nabla_h^2 \phi_R . \tag{163}$$

Here the acceleration potential ϕ_D is given by $g\bar{\bar{\eta}}$ in the simplest case in which one neglects in (55) all but the first term on the right side. In the more general case, however, the expression (162) has more the meaning of an abbreviation for the complex right side of (55) and the decomposition into two parts given by (162) is not possible in this case.

6.1 Vorticity equation

Writing the equation (41") in the following well-known local form (Serrin [27])

$$\frac{\partial \hat{\mathbb{V}}_h}{\partial t} - \hat{\mathbb{V}}_h \times (\nabla_h \times \hat{\mathbb{V}}_h + f\mathbf{k}) + \frac{\bar{\bar{P}}-\bar{\bar{E}}}{\bar{Q}} \hat{\mathbb{V}}_h = - \nabla_h \left(\phi_D + \frac{\hat{\mathbb{V}}_h^2}{2}\right) - \mathbf{k} \times \nabla_h \phi_R) \tag{41"'}$$

applying the curl-operation, and introducing the relative and absolute vorticity by

$$\hat{\zeta} = \mathbf{k} \cdot \nabla_h \times \hat{\mathbb{V}}_h , \quad \hat{\zeta}_a = \hat{\zeta} + f \tag{164}$$

one gets the vorticity equation in the following different forms:

$$\frac{\partial \hat{\zeta}}{\partial t} + \nabla_h \cdot \left\{ \hat{\zeta}_a \hat{\mathbb{V}}_h + \frac{\bar{\bar{P}}-\bar{\bar{E}}}{\bar{Q}} \hat{\mathbb{V}}_h \times \mathbf{k} \right\} = - \nabla_h^2 \phi_R \tag{165}$$

$$\frac{\hat{d}_h \hat{\zeta}_a}{dt} + \hat{\zeta}_a \nabla_h \cdot \hat{\mathbb{V}}_h + \mathbf{k} \cdot \nabla_h x \left\{ \frac{\overline{\overline{P}}-\overline{\overline{E}}}{\overline{Q}} \hat{\mathbb{V}}_h \right\} = -\nabla_h^2 \phi_R \tag{166}$$

or, after introduction of the equation of continuity

$$\nabla_h \cdot \hat{\mathbb{V}}_h = \overline{Q} \frac{\hat{d}_h}{dt} \left(\frac{1}{\overline{Q}}\right) + \frac{\overline{\overline{P}}-\overline{\overline{E}}}{\overline{Q}} \tag{167}$$

one obtains

$$\frac{\hat{d}_h}{dt} \left(\frac{\hat{\zeta}_a}{\overline{Q}} \right) + \frac{\hat{\zeta}_a}{\overline{Q}} \left(\frac{\overline{\overline{P}}-\overline{\overline{E}}}{\overline{Q}} \right) + \frac{1}{\overline{Q}} \mathbf{k} \cdot \nabla_h x \left\{ \frac{\overline{\overline{P}}-\overline{\overline{E}}}{\overline{Q}} \hat{\mathbb{V}}_h \right\} = -\frac{1}{\overline{Q}} \nabla_h^2 \phi_R \tag{168}$$

From equation (168), in the case $\overline{\overline{P}}-\overline{\overline{E}} = 0$, the simple vorticity equation

$$\frac{\hat{d}_h}{dt} \left(\frac{\hat{\zeta}_a}{\overline{Q}} \right) = -\frac{1}{\overline{Q}} \nabla_h^2 \phi_R \tag{169}$$

follows. This shows that only in the case $\nabla_h^2 \phi_R = 0$, i.e. the case in which only curl-free terms on the right side of (55) appear, a conservation law in the form

$$\frac{\hat{d}_h}{dt} \left(\frac{\hat{\zeta}_a}{\overline{Q}} \right) = 0 \tag{170}$$

exists (Charney [4], Morgan [20]). This is not the case however, in such complicated force fields as in the hurricane.

This can be seen by looking at the right side of equation (55) to which the lateral friction term on the left side is added in order to get $\mathbb{F}_h'$. Dividing by $\overline{Q}$ and taking the curl in general only the curl of the term $-g \nabla_h(\overline{\overline{\eta}}-\eta^{(T)})$ vanishes and therefore all other terms are acting as sources or sinks of the vorticity of the mean velocity vector $\hat{\mathbb{V}}_h$.

Some of the more essential parts of $-\nabla_h^2 \phi_R$ are given by

$$- \nabla_h^2 \phi_R = \mathbf{k} \cdot \nabla_h \times \frac{\mathbb{F}_h^{\nu}}{\bar{Q}} = \frac{1}{\bar{\bar{\rho}}^2} \mathbf{k} \cdot \nabla_h \bar{\bar{\rho}} \times \nabla_h \bar{\bar{p}}^{(a)} + \frac{1}{\bar{Q}} \mathbf{k} \cdot \nabla_h \times \left\{ \bar{\mathfrak{T}}_{z,h}^{(a)} - (\bar{\mathfrak{T}}_{z,h})_b \right\}$$

$$+ \frac{1}{\bar{Q}} \mathbf{k} \cdot \nabla_h \times (\nabla \cdot \bar{\mathcal{F}}_h) + \frac{g}{(\bar{\bar{\eta}}+H)^2} \mathbf{k} \cdot \nabla_h (\bar{\bar{\eta}}+H) \times \nabla_h \left(\frac{\overline{\eta'^2}}{2} \right)$$

$$- \frac{g}{2} \mathbf{k} \cdot \nabla_h \left\{ (\bar{\bar{\eta}}+H) + \frac{(\overline{\eta'^2})}{(\bar{\bar{\eta}}+H)} \right\} \times \nabla_h \ln \bar{\bar{\rho}} + \nabla_h \bar{\mathcal{E}}_b \times \nabla_h H$$

$$- \frac{1}{\bar{Q}^2} \mathbf{k} \cdot \nabla_h \bar{Q} \times \left\{ \bar{\mathfrak{T}}_{z,h}^{(a)} - (\bar{\mathfrak{T}}_{z,h})_b \right\} - \frac{1}{\bar{Q}^2} \mathbf{k} \cdot \nabla_h \bar{Q} \times (\nabla \cdot \bar{\mathcal{F}}_h)$$

$$+ \frac{g}{\bar{Q}^2} \mathbf{k} \cdot \nabla_h \bar{Q} \times \nabla_h \left(\overline{\int_{-H}^{\bar{\eta}} dz \int_z^{\eta} \bar{\rho}^* d\zeta} \right) + \frac{1}{\bar{Q}^2} \mathbf{k} \cdot \nabla_h \bar{Q} \times \nabla_h \overline{\mathcal{E}^{(i)}}$$

$$- \frac{1}{\bar{Q}} \mathbf{k} \cdot \nabla_h \times \overline{[(\mathfrak{T}'^{(a)}_{x,h} \mathbf{i} + \mathfrak{T}'^{(a)}_{y,h} \mathbf{j}) \cdot \nabla_h \bar{\eta}']} + \ldots \qquad (171)$$

where the dots denote further terms containing the density variations $\bar{\bar{\rho}}'$ and further terms which can be considered to be small.

An approximation found frequently in the literature is given by

$$- \nabla_h^2 \phi_R = \mathbf{k} \cdot \nabla_h \times \frac{\mathbb{F}_h^{\nu}}{\bar{Q}} \approx \frac{1}{\bar{Q}} \mathbf{k} \cdot \nabla_h \times \left\{ \bar{\mathfrak{T}}_{z,h}^{(a)} - (\bar{\mathfrak{T}}_{z,h})_b \right\} + \frac{1}{\bar{Q}} \mathbf{k} \cdot \nabla_h \times (\nabla \cdot \bar{\mathcal{F}}_h)$$

$$- \frac{g}{2} \mathbf{k} \cdot \nabla_h (\bar{\bar{\eta}}+H) \times \nabla_h \ln \bar{\bar{\rho}} \qquad (172)$$

which, however, appears mostly in connection with the vorticity equation for the mass transport vorticity $\mathbf{k} \cdot \nabla_h \times \mathbf{M}_h$ together with different assumptions regarding the vorticity of the lateral friction term.

Neglecting all but one term in (171), i.e. $- \nabla_h \left(\frac{1}{\bar{\bar{\rho}}} \right) \times \nabla_h \bar{\bar{p}}^{(a)}$ and also the term containing $\bar{\bar{P}}-\bar{\bar{E}}$ in (166) the equation

$$\frac{\hat{d}_h\hat{\zeta}_a}{dt} + \hat{\zeta}_a \nabla_h \cdot \hat{\mathbb{V}}_h = \nabla_h \bar{\bar{p}}^{(a)} x \nabla_h \left(\frac{1}{\bar{\bar{\rho}}}\right) \qquad (173)$$

has its analogy in the well-known vorticity equation for two-dimensional atmospheric motions where $\bar{\bar{p}}^{(a)}$, $\bar{\bar{\rho}}$, and $\hat{\zeta}$ have to be identified with the atmospheric pressure, density, and vorticity respectively. The case of "barotropic" flow (constant atmospheric pressure at the surface of the ocean or constant mean density $\bar{\bar{\rho}}$ of the ocean) leads to the barotropic vorticity equation in the simple case considered above and in the general case the first term in (171) drops out.

As already mentioned, the vorticity equation for the vorticity of the mass transport vector $\mathbf{M}_h$ cannot be expected to have such analogies to two-dimensional atmospheric motions in this simple manner because in atmospheric dynamics no vorticity equation for linear momentum has been studied systematically.

In order to get a vorticity equation for the vorticity of the mass transport vector, i.e.

$$z = \mathbf{k} \cdot \nabla_h \times \mathbf{M}_h = \mathbf{k} \cdot \nabla_h \times \bar{Q}\hat{\mathbb{V}}_h = \bar{Q}\hat{\zeta} + \mathbf{k} \cdot \nabla_h \bar{Q} \times \hat{\mathbb{V}}_h \qquad (174)$$

we write the basic equations (55') for the mass transport vector $\mathbf{M}_h$ in the form

$$\frac{\partial \mathbf{M}_h}{\partial t} + \nabla_h \cdot \left\{ \hat{\mathbb{V}}_h \mathbf{M}_h \right\} + f\mathbf{k} \times \mathbf{M}_h = \mathbb{F}_h$$

and introduce the identity

$$\nabla_h \cdot \left\{ \hat{\mathbb{V}}_h \mathbf{M}_h \right\} = z\mathbf{k} \times \hat{\mathbb{V}}_h + (\nabla_h \cdot \hat{\mathbb{V}}_h)\mathbf{M}_h + \nabla_h\left(\frac{\bar{Q}}{2}\hat{\mathbf{V}}_h^2\right) + \frac{\hat{\mathbf{V}}_h^2}{2} \nabla_h \bar{Q} \qquad (175)$$

leading to

$$\frac{\partial \mathbf{M}_h}{\partial t} + z\mathbf{k} \times \hat{\mathbb{V}}_h + (\nabla_h \cdot \hat{\mathbb{V}}_h) \mathbf{M}_h + f\mathbf{k} \times \mathbf{M}_h = \mathbb{F}_h - \nabla_h\left(\frac{\bar{Q}}{2}\hat{\mathbf{V}}_h^2\right) - \frac{\hat{\mathbf{V}}_h^2}{2} \nabla_h \bar{Q}$$

Applying the curl operation one gets

$$\frac{\partial Z}{\partial t} + \hat{\mathbb{V}}_h \cdot \nabla_h Z + 2Z(\nabla_h \cdot \hat{\mathbb{V}}_h) + f(\nabla_h \cdot \mathbf{M}_h) + \mathbf{M}_h \cdot \nabla_h f + \mathbf{k} \cdot \nabla_h (\nabla_h \cdot \hat{\mathbb{V}}_h) \times \mathbf{M}_h$$

$$= \mathbf{k} \cdot \nabla_h \times \mathbb{F}_h - \mathbf{k} \cdot \nabla_h \left(\frac{\hat{\mathbb{V}}_h^2}{2}\right) \times \nabla_h \bar{Q} \, . \tag{176}$$

The approximate form

$$\frac{\partial Z}{\partial t} + \mathbf{M}_h \cdot \nabla_h f = \mathbf{k} \cdot \nabla_h \times \mathbb{F}_h \tag{177}$$

has been used in several investigations in oceanography together with special assumptions on the structure of the force $\mathbb{F}_h$ (Sverdrup [32]).

The curl of $\mathbb{F}_h$ is given here by the curl of the right side of (55)

$$\begin{aligned}
\mathbf{k} \cdot \nabla_h \times \mathbb{F}_h = &- g\,\mathbf{k} \cdot \nabla_h \bar{Q} \times \nabla_h (\bar{\bar{\eta}} - \eta^{(T)}) - \mathbf{k} \cdot \nabla_h (\bar{\bar{\eta}} + H) \times \nabla_h \bar{\bar{p}}^{(a)} \\
&+ \mathbf{k} \cdot \nabla_h \times \left| \bar{\mathfrak{T}}^{(a)}_{z,h} - (\bar{\mathfrak{T}}_{z,h})_b \right| + \mathbf{k} \cdot \nabla_h \times (\nabla_h \cdot \bar{\mathfrak{F}}_h) \\
&- g\,\mathbf{k} \cdot \nabla_h \bar{\bar{\rho}} \times \nabla_h \left(\frac{\overline{\bar{\eta}'^2}}{2}\right) - \frac{g}{2}\mathbf{k} \cdot \nabla_h \left| (\bar{\bar{\eta}} + H)^2 + (\overline{\bar{\eta}'^2}) \right| \times \nabla_h \bar{\bar{\rho}} \\
&- \mathbf{k} \cdot \nabla_h \times \left\{ \overline{(\mathfrak{T}'^{(a)}_{x,h} \mathbf{i} + \mathfrak{T}'^{(a)}_{y,h} \mathbf{j}) \cdot \nabla_h \bar{\eta}'} \right\} + \mathbf{k} \cdot \nabla_h \bar{\mathcal{E}}_b \times \nabla_h H + \ldots
\end{aligned} \tag{178}$$

again neglecting small terms which, however, do not contain the following terms

$$- g \nabla_h \left(\overline{\int_{-H}^{\bar{\eta}} dz \int_z^{\bar{\eta}} \bar{\rho}^* d\zeta} \right); \quad - \nabla_h \overline{\mathcal{E}^{(i)}} \; ; \quad - g \nabla_h \left(\overline{\bar{\rho}' \frac{\bar{\eta}'^2}{2}}\right)$$

which are curl-free and vanish. Therefore the deviations $\bar{\rho}^*$ of the density from the vertical mean, the deviation of the vertical integrated pressure from the hydrostaticly calculated one, and the term $\overline{\bar{\rho}' \, \bar{\eta}'^2}$ do not contribute to the vorticity equation for the mass transport vorticity.

In the case of a homogeneous ocean we get

$$\mathbf{k}\cdot\nabla_h \times \mathbb{F}_h = -g\rho\mathbf{k}\cdot\nabla_h H \times \nabla_h \bar{\bar{\eta}} + g\rho\mathbf{k}\cdot\nabla_h(\bar{\bar{\eta}}+H) \times \nabla_h(\bar{\bar{\eta}}^{(a)}+\eta^{(T)})$$

$$+\mathbf{k}\cdot\nabla_h \times\left\{\overline{\mathbb{T}}^{(a)}_{z,h} - (\overline{\mathbb{T}}_{z,h})_b\right\} + \mathbf{k}\cdot\nabla_h \times(\nabla_h\cdot\overline{\mathcal{F}}_h) \tag{179}$$

$$-\mathbf{k}\cdot\nabla_h \times\left\{\overline{(\mathbb{T}'^{(a)}_{x,h}\mathbf{i}+\mathbb{T}'^{(a)}_{y,h}\mathbf{j})\cdot\nabla_h\bar{\eta}'}\right\} + \mathbf{k}\cdot\nabla_h\overline{\mathcal{E}}_b \times \nabla_h H + \ldots$$

6.2 Divergence equation

With respect to the derivation of the divergence equation we write (55') explicitly as

$$\frac{\partial\hat{\mathbb{V}}_h}{\partial t} + \hat{\mathbb{V}}_h\cdot\nabla_h\hat{\mathbb{V}}_h + f\mathbf{k}\times\hat{\mathbb{V}}_h + \frac{\bar{\bar{P}}-\bar{\bar{E}}}{\bar{Q}}\hat{\mathbb{V}}_h = -\nabla_h\phi_D - \mathbf{k}\times\nabla_h\phi_R \tag{55''}$$

and apply the divergence operation leading to

$$\frac{\partial}{\partial t}(\nabla_h\cdot\hat{\mathbb{V}}_h)+\hat{\mathbb{V}}_h\cdot\nabla_h(\nabla_h\cdot\hat{\mathbb{V}}_h)+\nabla_h\hat{\mathbb{V}}_h : \nabla_h\hat{\mathbb{V}}_h - f\hat{\zeta} - \hat{\mathbb{V}}_h\cdot(\mathbf{k}\times\nabla_h f)+\nabla_h\cdot\left(\frac{\bar{\bar{P}}-\bar{\bar{E}}}{\bar{Q}}\hat{\mathbb{V}}_h\right)$$
$$= -\nabla_h^2\phi_D$$

or, using the identity

$$\nabla_h\hat{\mathbb{V}}_h : \nabla_h\hat{\mathbb{V}}_h = (\nabla_h\cdot\hat{\mathbb{V}}_h)^2 - 2\mathcal{J}(\hat{V}_x,\hat{V}_y) = \mathcal{D}:\mathcal{D} - \frac{1}{2}\hat{\zeta}^2; \mathcal{D} = \frac{1}{2}\left|\nabla_h\hat{\mathbb{V}}_h + \hat{\mathbb{V}}_h\nabla_h\right|$$

where the absolute value of the deformation $\mathcal{D}$ tensor is given by $\sqrt{\mathcal{D}:\mathcal{D}}$

$$\left.\begin{aligned}
&\frac{\hat{d}_h}{dt}(\nabla_h\cdot\hat{\mathbb{V}}_h)+(\nabla_h\cdot\hat{\mathbb{V}}_h)^2 - 2\mathcal{J}(\hat{V}_x,\hat{V}_y) - f\hat{\zeta} - \hat{\mathbb{V}}_h\cdot(\mathbf{k}\times\nabla_h f)+\nabla_h\cdot\left(\frac{\bar{\bar{P}}-\bar{\bar{E}}}{\bar{Q}}\hat{\mathbb{V}}_h\right) = -\nabla_h^2\phi_D\\
&\frac{\hat{d}_h}{dt}(\nabla_h\cdot\hat{\mathbb{V}}_h)+\mathcal{D}:\mathcal{D} - \frac{1}{2}\hat{\zeta}_a^2 - \hat{\mathbb{V}}_h\cdot(\mathbf{k}\times\nabla_h f)+\nabla_h\cdot\left(\frac{\bar{\bar{P}}-\bar{\bar{E}}}{\bar{Q}}\hat{\mathbb{V}}_h\right) = -\left(\nabla_h^2\phi_D + \frac{f^2}{2}\right)
\end{aligned}\right\}$$

(180)

The two forms of the divergence equation have, of course, their analogy in atmospheric dynamics, where only the right hand side again has a different physical meaning.

Since $\mathcal{D}:\mathcal{D}$ and $\hat{\zeta}_a^2$ are always greater than zero the following inequality can be derived from the second representation of the divergence equation (Truesdell and Toupin [34]):

$$\frac{1}{2}\hat{\zeta}_a^2 \geqq (\nabla_h^2 \phi_D + \frac{f^2}{2}) + \frac{\hat{d}_h}{dt}(\nabla_h \cdot \hat{\mathbb{V}}_h) - \hat{\mathbb{V}}_h \cdot (\mathbf{k} \times \nabla_h f) + \nabla_h \cdot (\frac{\overline{\overline{P}}-\overline{\overline{E}}}{\overline{Q}} \hat{\mathbb{V}}_h) \geqq -\mathcal{D}:\mathcal{D} \tag{181}$$

This inequality gives an upper and a lower limit for the expression within the two inequality signs, essentially for the divergence of the negative accelerations $-\frac{\mathbb{F}_h}{\overline{Q}}$ or, for the Laplacian of the acceleration potential ϕ_D, if the other terms are sufficiently small.

This apparently useful approximation is given by

$$\frac{1}{2}\hat{\zeta}_a^2 \geqq \nabla_h^2 \phi_D + \frac{f^2}{2} \geqq -\mathcal{D}:\mathcal{D} \tag{182}$$

where, however, the limits are unknown functions. Therefore in contrast to the conditions in atmospheric dynamics where the absolute vorticity and the deformation can be obtained from wind observations, the divergence equation in this form seems to have little value for practical applications to storm surge problems.

The divergence equation for mass transport, however, which has not been used in an explicit form in order to derive the wave equation (90'), probably has more importance than (180), mainly with respect to certain approximations.

Starting again with equation (55'), i.e.

$$\frac{\partial \mathbf{M}_h}{\partial t} + \nabla_h \cdot \left\{ \hat{\mathbb{V}}_h \mathbf{M}_h \right\} + f\mathbf{k} \times \mathbf{M}_h = \mathbb{F}_h$$

and applying the divergence operation one gets, using the identity

$$\nabla_h \cdot \left\{ \hat{\mathbb{V}}_h \mathbf{M}_h \right\} = \frac{1}{\overline{Q}} \mathcal{D}_M : \mathcal{D}_M - \frac{1}{2}\frac{Z^2}{\overline{Q}} + \hat{\mathbb{V}}_h \cdot \nabla_h(\nabla_h \cdot \mathbf{M}_h) + \mathbf{M}_h \cdot \nabla_h(\nabla_h \cdot \hat{\mathbb{V}}_h)$$
$$+ (\nabla_h \cdot \hat{\mathbb{V}}_h)(\nabla_h \cdot \mathbf{M}_h) + \hat{\mathbb{V}}_h \cdot (\nabla_h \mathbf{M}_h) \cdot \nabla_h (\frac{1}{\overline{Q}}) \tag{183}$$

the divergence equation for the divergence of the mass transport in the form

$$\frac{\hat{d}_h}{dt}(\nabla_h \cdot \mathbf{M}_h) + \frac{1}{\overline{Q}} \mathcal{D}_M : \mathcal{D}_M - \frac{1}{2}\frac{Z^2}{\overline{Q}} + \mathbf{M}_h \cdot \nabla_h(\nabla_h \cdot \hat{\mathbb{V}}_h) + (\nabla_h \cdot \hat{\mathbb{V}}_h)(\nabla_h \cdot \mathbf{M}_h)$$
$$+ \hat{\mathbb{V}}_h \cdot (\nabla_h \mathbf{M}_h) \cdot \nabla_h (\frac{1}{\overline{Q}}) - fZ - \mathbf{M}_h \cdot (\mathbf{k} \times \nabla_h f) = \nabla_h \cdot \mathbb{F}_h \tag{184}$$

where $\mathcal{D}_M$ is the deformation tensor of the mass transport field $\mathbf{M}_h$.

In a linear theory only the following divergence equation appears

$$\frac{\partial}{\partial t}(\nabla_h \cdot \mathbf{M}_h) - fZ - \mathbf{M}_h \cdot (\mathbf{k} \times \nabla_h f) = \nabla_h \cdot \mathbb{F}^{\nu}_h \tag{185}$$

and therefore neglects all products of velocities or mass transports and their derivatives.

The corresponding linearized vorticity equation follows from (176)

$$\frac{\partial Z}{\partial t} + f(\nabla_h \cdot \mathbf{M}_h) + \mathbf{M}_h \cdot \nabla_h f = \mathbf{k} \cdot \nabla_h \times \mathbb{F}^{\nu}_h \tag{186}$$

This together with (185) and the equation of continuity (41') builds up a system from which, in the case $\nabla_h f = 0$, a wave equation can be derived by eliminating the vorticity and the divergence. This leads to a simplified form of (90'); this equation was derived in the same way from the vorticity and divergence equation.

There remains the calculation of the divergence of the force field $\mathbb{F}^{\nu}_h$ which is given by

$$\begin{aligned}
\nabla_h \cdot \mathbb{F}^{\nu}_h = & - g\bar{Q}\nabla_h^2(\bar{\bar{\eta}} - \eta^{(T)}) - (\bar{\bar{\eta}} + H)\nabla_h^2 \bar{\bar{p}}^{(a)} + \nabla_h \cdot \left| \overline{\mathcal{T}}^{(a)}_{z,h} - (\overline{\mathcal{T}}_{z,h})_b \right| + \nabla_h \cdot (\nabla_h \cdot \overline{\mathcal{F}}_h) \\
& - g\nabla_h \bar{Q} \cdot \nabla_h(\bar{\bar{\eta}} - \eta^{(T)}) - \nabla_h (\bar{\bar{\eta}} + H) \cdot \nabla_h \bar{\bar{p}}^{(a)} \\
& - g\,\bar{\bar{\bar{\rho}}}\nabla_h^2 \left(\frac{\overline{\bar{\eta}'^2}}{2} \right) - \frac{g}{2} \left\{ (\bar{\bar{\eta}} + H)^2 + (\overline{\bar{\eta}'^2}) \right\} \nabla_h^2 \bar{\bar{\bar{\rho}}} - g\nabla_h^2 \left(\overline{\int_{-H}^{\bar{\eta}} dz \int_z^{\bar{\eta}} \bar{\rho}^* d\zeta} \right) \\
& - 2g\nabla_h \bar{\bar{\bar{\rho}}} \cdot \nabla_h \left(\frac{\overline{\bar{\eta}'^2}}{2} \right) - g(\bar{\bar{\eta}} + H)\nabla_h(\bar{\bar{\eta}} + H) \cdot \nabla_h \bar{\bar{\bar{\rho}}} - \nabla_h \cdot \left\{ \overline{(\mathcal{T}'^{(a)}_{x,h} \mathbf{i} + \mathcal{T}'^{(a)}_{y,h} \mathbf{j}) \cdot \nabla_h \bar{\eta}'} \right\} \\
& + \overline{\mathcal{E}_b Q}\nabla_h^2 H + \nabla_h (\overline{\mathcal{E}_b Q}) \cdot \nabla_h H - \nabla_h^2 \overline{\mathcal{E}^{(i)}} + \ldots
\end{aligned} \tag{187}$$

This follows from the right side of (55) together with the lateral friction term from the left side and in which the remaining small terms have not been written down explicitly.

It can be seen, that even in the case of a homogeneous ocean the Laplacian of the variance $\left(\frac{\overline{\bar{\eta}'^2}}{2}\right)$ and also of the non-hydrostatic part of the averaged pressure enters the divergence equations (180) as well as the divergence equation (184).

If we can assume a curl-free mass transport, equation (185) (with $\nabla_h f \approx 0$) together with (187) for a homogeneous ocean with flat bottom and $\bar{\bar{\eta}} \ll H$ leads to the well-known wave equation (with added surface wave effects)

$$\frac{\partial^2 \bar{\bar{\eta}}}{\partial t^2} = gH\nabla_h^2 (\bar{\bar{\eta}} - \bar{\bar{\eta}}^{(a)}) + \frac{1}{\rho}\nabla_h \cdot \left\{ \overline{\mathfrak{T}}_{z,h}^{(a)} - (\overline{\mathfrak{T}}_{z,h})_b \right\} + g\nabla_h^2 \left(\frac{\overline{\eta'^2}}{2} \right)$$

$$+ \frac{1}{\rho}\nabla_h \cdot \left\{ \overline{(\mathfrak{T}'^{(a)}_{x,h}\,\mathbf{i} + \mathfrak{T}'^{(a)}_{y,h}\,\mathbf{j}) \cdot \nabla_h \, \bar{\eta}'} \right\} \qquad (188)$$

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